Matrix Solutions to Linear Systems

Objective: To solve systems using matrix equations.

Review
Solve this system of equations.

\[4x + 2y = -3\]
\[x + 5y = -4\]

Systems of Equations in Two Variables

Matrices

A rectangular array of numbers is called a _______(plural, matrices).

The ________ of a matrix are vertical.

The matrix shown has 2 rows and 3 columns.

A matrix with \(m\) rows and \(n\) columns is said to be of order __________.

When \(m = n\) the matrix is said to be square.

See Example 1, page 783.

Matrices

Consider the system:

\[4x + 2y = -3\]
\[x + 5y = -4\]

Example:

\[
\begin{bmatrix}
4 & 2 & -3 \\
1 & 5 & -4 \\
\end{bmatrix}
\]

1) Interchange any two _______. \((R_i \leftrightarrow R_j)\)
2) ________each entry in a row by the same nonzero constant. \((kR_i)\)
3) ________ a nonzero multiple of one row to another row. \((kR_i + R_j)\)

Row-Equivalent Operations
Example

Perform each indicated row operation on:

a) \((R_1 \rightarrow R_2)\)

\[
\begin{bmatrix}
2 & -1 & 1 & 6 \\
1 & -3 & -2 & -1 \\
4 & 0 & 5 & 23
\end{bmatrix}
\]

b) \(\frac{1}{4}R_1\)

c) \(3R_2 + R_3 = \text{new } R_3\)

Example – Watch and listen.

Solve the following system:

\[
\begin{align*}
2x - y + z &= 8 \\
x - 3y - 2z &= -1 \\
4x + 5z &= 23
\end{align*}
\]

Example continued

We multiply the first row by \(-2\) and add it to the second row.
We also multiply the first row by \(-4\) and add it to the third row.

\[
\begin{bmatrix}
2 & -1 & 1 & 8 \\
1 & -3 & -2 & -1 \\
4 & 0 & 5 & 23
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & -3 & -2 & -1 \\
2 & -1 & 1 & 8 \\
4 & 0 & 5 & 23
\end{bmatrix}
\]

\(R_1 \leftrightarrow R_2\)

We multiply the second row by \(1/5\) to get a 1 in the second row, second column.

\[
\begin{bmatrix}
1 & -3 & -2 & -1 \\
0 & 5 & 5 & 10 \\
0 & 12 & 13 & 27
\end{bmatrix}
\]

Solving Systems using Gaussian Elimination with Matrices

1. Write the _________ matrix.
2. Use row operations to get the matrix in “row echelon” form:

\[
\begin{bmatrix}
a & b & c \\
d & e & f \\
0 & 0 & 1
\end{bmatrix}
\]

3. Write the _________ of equations corresponding to the resulting matrix.
4. Use back-substitution to find the system’s solution.

First, we write the augmented matrix, writing 0 for the missing \(y\)-term in the last equation.

\[
\begin{bmatrix}
2 & -1 & 1 & 8 \\
1 & -3 & -2 & -1 \\
4 & 0 & 5 & 23
\end{bmatrix}
\]

Our goal is to find a row-equivalent matrix of the form

\[
\begin{bmatrix}
a & b & c \\
0 & 0 & 1
\end{bmatrix}
\]

We multiply the second row by \(1/5\) to get a 1 in the second row, second column.

\[
\begin{bmatrix}
1 & -3 & -2 & -1 \\
0 & 5 & 5 & 10 \\
0 & 12 & 13 & 27
\end{bmatrix}
\]

\(R_2 = \frac{1}{5}R_2\)
We multiply the second row by \(-12\) and add it to the third row.

\[
\begin{bmatrix}
1 & -3 & -2 & -1 \\
0 & 1 & 1 & 2 \\
0 & 12 & 13 & 27 \\
\end{bmatrix} =
\begin{bmatrix}
1 & -3 & -2 & -1 \\
0 & 1 & 1 & 2 \\
0 & 0 & 1 & 3 \\
\end{bmatrix}
R_3 = -12R_2 + R_3
\]

Example continued

\[\begin{bmatrix}
1 & -3 & -2 & -1 \\
0 & 1 & 1 & 2 \\
0 & 0 & 1 & 3 \\
\end{bmatrix}
\]

\[\begin{array}{l}
\text{Now, we can write the system of} \\
equations that corresponds to the last matrix:
\end{array}\]

\[x - 3y - 2z = -1\]
\[y + z = 2\]
\[z = 3\]

Example continued

\[\begin{array}{l}
\text{We back-substitute 3 for} \\
z \text{ in equation (2) and solve for } y.
\end{array}\]

\[y + z = 2\]
\[y + 3 = 2\]
\[y = -1\]

Example

\[\begin{array}{l}
\text{Use matrices to solve the system}
\end{array}\]

\[2x + y + 2z = 18\]
\[x - y + 2z = 9\]
\[x + 2y - z = 6.\]

1. Write the augmented matrix for the system.
2. Solve using Gauss Elimination or your calculator.
3. Check/verify your solution.

Example

\[\begin{array}{l}
\text{2x + y + 2z = 18}
\end{array}\]
\[\begin{array}{l}
x - y + 2z = 9
\end{array}\]
\[\begin{array}{l}
x + 2y - z = 6.
\end{array}\]

\[\begin{array}{l}
x = 2
\end{array}\]

\[\begin{array}{l}
\text{The triple (2, -1, 3) checks in the original system of equations, so it is the solution.}
\end{array}\]
Row-Echelon Form

1. If a row does not consist entirely of 0’s, then the first nonzero element in the row is a 1 (called a leading 1).
2. For any two successive nonzero rows, the leading 1 in the lower row is farther to the _________ than the leading 1 in the higher row.
3. All the rows consisting entirely of 0’s are at the _________ of the matrix.
   If a fourth property is also satisfied, a matrix is said to be in reduced row-echelon form:
4. Each column that contains a leading 1 has 0’s everywhere else.

Example -- Which of the following matrices are in row-echelon form?

a)
\[
\begin{bmatrix}
1 & -6 & 7 & 5 \\
0 & 1 & -3 & 4 \\
0 & 0 & 1 & 8 \\
\end{bmatrix}
\]

b)
\[
\begin{bmatrix}
0 & -2 & 4 & 1 \\
0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

c)
\[
\begin{bmatrix}
1 & -2 & -7 & 6 \\
0 & 0 & 0 & 0 \\
0 & 1 & 4 & -2 \\
\end{bmatrix}
\]

d)
\[
\begin{bmatrix}
1 & 0 & 0 & -3.5 \\
0 & 1 & 0 & 0.7 \\
0 & 0 & 1 & 4.5 \\
\end{bmatrix}
\]

Gauss-Jordan Elimination

□ We perform row-equivalent operations on a matrix to obtain a row-equivalent matrix in row-echelon form. We continue to apply these operations until we have a matrix in reduced row-echelon form.

Example: Use Gauss-Jordan elimination to solve the system of equations from the previous example; we had obtained the matrix
\[
\begin{bmatrix}
1 & -3 & -2 & -1 \\
0 & 1 & 1 & 2 \\
0 & 0 & 1 & 3 \\
\end{bmatrix}
\]

Gauss-Jordan Elimination continued -- We continue to perform row-equivalent operations until we have a matrix in reduced row-echelon form.

\[
\begin{bmatrix}
1 & -3 & 0 & 5 \\
0 & 1 & 0 & -1 \\
0 & 0 & 1 & 3 \\
\end{bmatrix}
\]

□ Next, we multiply the second row by 3 and add it to the first row.
\[
\begin{bmatrix}
1 & 0 & 0 & 2 \\
0 & 1 & 0 & -1 \\
0 & 0 & 1 & 3 \\
\end{bmatrix}
\]

□ Writing the system of equations that corresponds to this matrix, we have
\[
\begin{align*}
x &= 2 \\
y &= -1 \\
z &= 3
\end{align*}
\]

□ We can actually read the solution, \((2, -1, 3)\), directly from the last column of the reduced row-echelon matrix.
When a row consists entirely of 0’s, the equations are ________ and the system is equivalent, meaning you have a system that is colinear.

Answer: INFINITELY MANY SOLUTIONS

When we obtain a row whose only nonzero entry occurs in the last column, we have an inconsistent system of equations. For example, in the matrix

$$\begin{bmatrix}
1 & 0 & 4 & -6 \\
0 & 1 & 4 & 8 \\
0 & 0 & 0 & 9
\end{bmatrix}$$

the last row corresponds to the equation $0 = 9$, so we know the original system has no solution.

Example

Solve the system:

$$\begin{align*}
x - 2y - z &= 5 \\
2x - 5y + 3z &= 6 \\
x - 3y + 4z &= 1
\end{align*}$$