3.5: Exponential Growth and Decay; Modeling Data

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Outline

1. Exponential Growth and Decay; Modeling Data
   - Exponential Growth and Decay Models
   - Half Life
Exponential Growth and Decay Models

The model for exponential growth or decay is given by:

$$A = A_0 e^{kt},$$

where
- $A_0$ - original amount,
- $t$ - time,
- $k$ - rate,
- $A$ - amount (after $t$).

If $k > 0$, that is, if $k$ is a positive number, the model represents a growth. If $k < 0$, that is, if $k$ is a negative number, the model represents a decay.
Exponential Growth and Decay Models

\[ A = A_0 e^{kt}, \ k > 0 \]

Exponential growth

\[ A = A_0 e^{kt}, \ k < 0 \]

Exponential decay
Example

1. \( A = 401e^{0.003t} \) represents the population, \( A \), of a country in millions, \( t \) years after 2001. Find the population in 2001.

Since the time is measured starting from 2001, the year 2001 represents \( t = 0 \). To find the population in 2001, replace \( t \) with 0 in the above function, and find \( A \).

\[
A = 401e^{0.003t} \\
= 401e^{0.003 \cdot 0} \\
= 401e^0 \\
= 401 \cdot 1 \\
= 401
\]

The given function

Replace \( t \) with 0

Simplify

\( e^0 = 1 \)

Simplify

The population in 2001 was 401 million.
Example

2. A = 451e^{0.002t} represents the population, A, of a country in millions, t years after 2003. Find the population in 2009. (Round to two decimal places.)

Since the time is measured starting from 2003, the year 2009 represents t = 6. To find the population in 2009, replace t with 6 in the above function, and find A.

\[ A = 451e^{0.002t} \]
\[ = 451e^{0.002 \cdot 6} \]
\[ = 451e^{0.012} \]
\[ \approx 456.44 \]

The population in 2009 was 456.44 million.
Example

3. $A = 170e^{0.007t}$ represents the population, $A$, of a country in millions, $t$ years after 2004. When will be the population 200 million?

Replace $A$ with 200, and find $t$.

\[
\begin{align*}
A &= 170e^{0.007t} \\
200 &= 170e^{0.007t} \\
\frac{200}{170} &= e^{0.007t} \\
\ln \left( \frac{200}{170} \right) &= \ln \left( e^{0.007t} \right) \\
\ln \left( \frac{200}{170} \right) &= 0.007t \cdot \ln(e) \\
\ln \left( \frac{200}{170} \right) &= 0.007t \\
\ln e &= 1
\end{align*}
\]
Example

\[ A = 170e^{0.007t} \]

\[ 200 = 170e^{0.007t} \]

\[ \frac{200}{170} = e^{0.007t} \]

\[ \ln \left( \frac{200}{170} \right) = \ln \left( e^{0.007t} \right) \]

\[ \ln \left( \frac{200}{170} \right) = 0.007t \cdot \ln(e) \]

\[ \ln \left( \frac{200}{170} \right) = 0.007t \]

\[ \frac{\ln \left( \frac{200}{170} \right)}{0.007} = t \]

\[ t \approx 23 \]

The given function

Replace \( A \) with 200

Divide by 170

Take \( \ln \) of both sides

Simplify

\( \ln(e) = 1 \)

Divide by 0.007

The population will be 200 million in 2027 (2004 + 23 years).
Example

4. In 2000, the population of a country was 5.7 million, and by 2027 it is projected to grow to 8 million. Find an exponential growth function \( A = A_0 e^{kt} \) that models the data, where \( t \) is the number of years after 2000. When will be the population 12 million?

Since the time is measured starting from 2000, \( A_0 = 5.7 \) represents the initial population. First write the function using \( k \) as the growth rate. Then find \( k \) using \( A = 8 \) and \( t = 27 \).

\[
A = 5.7e^{kt} \quad \text{The given function}
\]

\[
8 = 5.7e^{k \cdot 27} \quad \text{Replace } A \text{ with } 8 \text{ and } t \text{ with } 27
\]

\[
\frac{8}{5.7} = e^{27k} \quad \text{Divide by } 5.7
\]

\[
\ln \left( \frac{8}{5.7} \right) = \ln \left( e^{27k} \right) \quad \text{Take ln of both sides}
\]
Example

Since the time is measured starting from 2000, \( A_0 = 5.7 \) represents the initial population. First write the function using \( k \) as the growth rate. Then find \( k \) using \( A = 8 \) and \( t = 27 \).

\[
A = 5.7e^{kt} \quad \text{The given function}
\]

\[
8 = 5.7e^{k \cdot 27} \quad \text{Replace } A \text{ with } 8 \text{ and } t \text{ with } 27
\]

\[
\frac{8}{5.7} = e^{27k} \quad \text{Divide by } 5.7
\]

\[
\ln \left( \frac{8}{5.7} \right) = \ln \left( e^{27k} \right) \quad \text{Take } \ln \text{ of both sides}
\]

\[
\ln \left( \frac{8}{5.7} \right) = 27k \cdot \ln(e) \quad \text{Simplify}
\]

\[
\ln \left( \frac{8}{5.7} \right) = 27k \quad \ln(e) = 1
\]
Example

\[ \frac{8}{5.7} = e^{27k} \]

Divide by 5.7

\[ \ln \left( \frac{8}{5.7} \right) = \ln (e^{27k}) \]

Take \ln of both sides

\[ \ln \left( \frac{8}{5.7} \right) = 27k \cdot \ln(e) \]

Simplify

\[ \ln \left( \frac{8}{5.7} \right) = 27k \]

\[ \ln e = 1 \]

\[ \ln \left( \frac{8}{5.7} \right) = 27k \]

\[ \frac{27}{27} = k \]

Divide by 27

\[ k \approx 0.0126 \]

Approximate

Replace \( k \) with 0.0126 in the function \( A = 5.7e^{kt} \).
Example

Replace \( k \) with 0.0126 in the function \( A = 5.7e^{kt} \). Then find \( t \) using \( A = 12 \).

\[
A = 5.7e^{0.0126t} \\
12 = 5.7e^{0.0126t} \\
\frac{12}{5.7} = e^{0.0126t} \\
\ln\left(\frac{12}{5.7}\right) = \ln\left(e^{0.0126t}\right) \\
\ln\left(\frac{12}{5.7}\right) = 0.0126t \cdot \ln(e) \\
\ln\left(\frac{12}{5.7}\right) = 0.0126t \\
\ln\left(\frac{12}{5.7}\right) = t \\
\frac{\ln\left(\frac{12}{5.7}\right)}{0.0126} = t
\]

The given function
Replace \( A \) with 12
Divide by 5.7
Take \( \ln \) of both sides
Simplify
\( \ln e = 1 \)
Divide by 0.0126
Example

\[ A = 5.7e^{0.0126t} \]
\[ 12 = 5.7e^{0.0126t} \]
\[ \frac{12}{5.7} = e^{0.0126t} \]
\[ \ln \left( \frac{12}{5.7} \right) = \ln \left( e^{0.0126t} \right) \]
\[ \ln \left( \frac{12}{5.7} \right) = 0.0126t \cdot \ln(e) \]
\[ \ln \left( \frac{12}{5.7} \right) = 0.0126t \]
\[ \ln \left( \frac{12}{5.7} \right) = t \]
\[ t \approx 59 \]

The given function
Replace \( A \) with 12
Divide by 5.7
Take ln of both sides
Simplify
\( \ln e = 1 \)
Divide by 0.0126
Approximate

The population will be 12 million in 2059 (2000 + 59 years).
Example

5. The half-life of an element is 5 seconds. If 128 grams are initially present, how many grams are present after 5 seconds? 10 seconds? 15 seconds? 20 seconds? 25 seconds?

Since the half-life is 5 seconds, every 5 seconds, the amount becomes half of the present amount.

The amount after 5 seconds is 64 grams. Half of 128
The amount after 10 seconds is 32 grams. Half of 64
The amount after 15 seconds is 16 grams. Half of 32
The amount after 20 seconds is 8 grams. Half of 16
The amount after 25 seconds is 4 grams. Half of 8