Exponential Growth & Decay

Objective: To apply models of exponential growth and decay.

Exponential Growth & Decay Models

\[ A = A_0 e^{kt} \]

- \( A_0 \) is the amount you start with, \( t \) = time, and \( k \) = constant of growth (or decay)
- If \( k > 0 \), the amount is growing (getting larger), as in the money in a savings account that is having interest compounded over time
- If \( k < 0 \), the amount is shrinking (getting smaller), as in the amount of radioactive substance remaining after the substance decays over time

Graphs

Could the following graph model exponential growth or decay?

- a) Growth model.
- b) Decay model.

Population Growth

- The function \( A = A_0 e^{kt}, \ k > 0 \) can model many kinds of population growths.

In this function:
- \( A_0 \) = population at time 0,
- \( A \) = population after time,
- \( t \) = amount of time,
- \( k \) = exponential growth rate.

The growth rate unit must be the same as the time unit.

Example – Watch and listen.

- Population Growth of the United States.
  In 1990 the population in the United States was about 249 million and the exponential growth rate was 8% per decade. (Source: U.S. Census Bureau)
  - Find the exponential growth function.
  - What will the population be in 2020?
  - After how long will the population be double what it was in 1990?
Solution:
Find the exponential growth function.

At \( t = 0 \) (1990), the population was about 249 million. We substitute 249 for \( A_0 \) and 0.08 for \( k \) to obtain the exponential growth function.

\[
P(t) = 249e^{0.08t}
\]

Solution:
What will the population be in 2020?

In 2020, 3 decades later, \( t = 3 \). To find the population in 2020 we substitute 3 for \( t \):

\[
P(3) = 249e^{0.08(3)} = 249e^{0.24} \approx 317.
\]

The population will be approximately 317 million in 2020.

Example 1

In 1990, the population of Africa was 643 million and by 2000 it had grown to 813 million.

a) Use the exponential growth model \( A = A_0e^{kt} \) in which \( t \) is number of years after 1990, to find the exponential growth function that models the data.

b) By which year will Africa’s population reach 2000 million, or two billion?

Exponential Decay

Decay, or decline, of a population is represented by the function \( A = A_0e^{-kt} \), \( k > 0 \).

In this function:

- \( A_0 \) = initial amount of the substance,
- \( A \) = amount of the substance left after time,
- \( t \) = time,
- \( k \) = decay rate.

The half-life is the amount of time it takes for half of an amount of substance to decay.

Example 2

Carbon Dating. The radioactive element carbon-14 has a half-life of 5715 years. The half-life of a substance is the time required for half of the substance to disintegrate.

The function for carbon dating is \( A = A_0e^{-0.000121t} \).

Skeletons were found at a construction site in San Francisco in 1989. The skeletons contained 88% of the expected amount of carbon-14 found in a living person. In 1989, how old were the skeletons?
Example 3

- Strontium-90 is a waste product from nuclear reactors. As a consequence of fallout from atmospheric nuclear tests, we all have a measurable amount of strontium-90 in our bones.
- a) The half-life of strontium-90 is 28 years. Find the exponential decay model for strontium-90.
- b) Suppose that a nuclear accident occurs and releases 60 grams of strontium-90 into the atmosphere. How long will it take for strontium-90 to decay to a level of 10 grams?

Logistic Growth Model

Logistic Growth Model

The mathematical model for limited logistic growth is given by:

\[ f(t) = \frac{c}{1 + ae^{-bt}} \quad \text{or} \quad A = \frac{c}{1 + ae^{-bt}} \]

where \( a, b, \) and \( c \) are constants, with \( c > 0 \) and \( b > 0 \).

Example 4

- The logistic growth function below models the percentage, \( P(x) \), of Americans who are \( x \) years old with some coronary heart disease.

\[ P(x) = \frac{90}{1 + 271e^{-0.122x}} \]

- At what age is the percentage of some coronary heart disease 70%?