Inverse Functions

Section 1.8

Objectives

– Verify inverse functions
– Use the horizontal line test to determine if a function is a one-to-one function.
– Find the inverse of a function.
– Given a graph, graph the inverse.
– Find the inverse of a function & graph both functions simultaneously.

What is an inverse function?

• A function that "undoes" the original function.

• A function "wraps an x" and the inverse would "unwrap the x" resulting in x when the 2 functions are composed on each other.

\[ f(f^{-1}(x)) = f^{-1}(f(x)) = x \]

Example

Given that \( f(x) = 7x - 2 \), use composition of functions to show that \( f^{-1}(x) = \frac{x + 2}{7} \).

Do all functions have inverses?

• Yes, and no. Yes, they all will have inverses, BUT we are only interested in the inverses if they ARE A FUNCTION.

• DO ALL FUNCTIONS HAVE INVERSES THAT ARE FUNCTIONS? NO.

• Recall, functions must pass the vertical line test when graphed. If the inverse is to pass the vertical line test, the original function must pass the HORIZONTAL line test (be one-to-one!)

One-to-One Functions

A function \( f(x) \) is a one-to-one function if x-values do not share the same y-values.

Remember that a function will have different x-values.
A one-to-one function will have different x-values and different y-values.
Why are one-to-one functions important?

One-to-One Functions have Inverse functions

Horizontal Line Test

- Use to determine whether a function is one-to-one.
- A function is one-to-one if and only if no horizontal line intersects its graph more than once.

Horizontal-Line Test

Graph \( f(x) = -3x + 4 \).

Example: From the graph at the left, determine whether the function is one-to-one and thus has an inverse that is a function.

Horizontal-Line Test

Graph \( f(x) = x^2 - 2 \).

Example: From the graph at the left, determine whether the function is one-to-one and thus has an inverse that is a function.

How do you find an inverse?

- "Undo" the function.
- Replace the \( x \) with \( y \) and solve for \( y \).

How to find the Inverse of a One-to-One Function

1. Replace \( f(x) \) with \( y \) in the equation.
2. Interchange \( x \) and \( y \) in the equation.
3. Solve this equation for \( y \).
4. Replace \( y \) with \( f^{-1}(x) \).

Any restrictions on \( x \) or \( y \) should be considered and included with the equation.

Remember: Domain and Range are interchanged for inverses.
Example
Determine whether the function \( f(x) = 3x - 2 \) is one-to-one, and if it is, find a formula for \( f^{-1}(x) \).

Graph of Inverse f⁻¹ function
• The graph of \( f^{-1} \) is obtained by reflecting the graph of \( f \) across the line \( y = x \).

Properties of One-to-One Functions and Inverses
• If a function is one-to-one, then its inverse is a function.
• The domain of a one-to-one function \( f \) is the range of the inverse \( f^{-1} \).
• The range of a one-to-one function \( f \) is the domain of the inverse \( f^{-1} \).
• A function that is increasing over its domain or is decreasing over its domain is a one-to-one function.

Solution
<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) = 3x - 2 )</th>
<th>( f^{-1}(x) = \frac{x+2}{3} )</th>
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Example
Graph \( f(x) = 3x - 2 \) and \( f^{-1}(x) = \frac{x+2}{3} \) using the same set of axes.
Then compare the two graphs.
Determine the domain and range of the function and its inverse.

Example
Graph \( y = \frac{1}{3}x + 2 \) and its inverse \( 3(x-2) \).
Every point on the graph \( (x,y) \) exists on the inverse as \( (y,x) \) (i.e., if \((-6,0)\) is on the graph, \((0,-6)\) is on its inverse.

How do their graphs compare?
• The graph of a function and its inverse always mirror each other through the line \( y=x \).
Restricting a Domain

- When the inverse of a function is *not* a function, the domain of the function can be restricted to allow the inverse to be a function.
- In such cases, it is convenient to consider "part" of the function by restricting the domain of $f(x)$. If the domain is restricted, then its inverse is a function.

Restricting the Domain

Recall that if a function is not one-to-one, then its inverse will not be a function.

If we restrict the domain values of $f(x)$ to those greater than or equal to zero, we see that $f(x)$ is now one-to-one and its inverse is now a function.