Chapter 6
Rational Number Operations and Properties

Section 6.3
Multiplying and Dividing Fractions

The “Whole” as a Collection of Objects
- Thus far, we’ve mostly discussed fractions of wholes where the wholes can be represented by a “continuous” quantity (which we represent by an area).
- We can also talk about fractions of wholes where the whole is a collection of objects.
- We have in fact encountered this but haven’t described it in this kind of language.

Revisiting the Brownies
- Consider the problem “If we share 7 brownies fairly among 3 people, how much brownie does each person get?”
- We are dividing the collection of 7 brownies into 3 equal shares. Each person gets one of those shares. This means that each person gets \( \frac{1}{3} \) of the collection of brownies, i.e. \( \frac{1}{3} \) of 7 brownies. We have seen that \( \frac{1}{3} \) of 7 brownies (one share) = \( \frac{7}{3} \) brownies.

Identify the Wholes
- 7 brownies implies that we have 7 objects where each object is one brownie (whole = 1 brownie).
- \( \frac{2}{3} \) brownies implies that \( \frac{2}{3} \) refers to the whole of 1 brownie.
- However, \( \frac{2}{3} \) of 7 brownies implies that \( \frac{1}{3} \) refers to the whole of 7 brownies (i.e. the collection of 7 brownies).
- What would \( \frac{2}{3} \) of 7 brownies mean?
- Going by the definition of the fraction \( \frac{2}{3} \), we divide the whole (collection of 7 brownies) into 3 equal shares and choose 2 of those shares. [The answer would be \( \frac{2}{3} + \frac{1}{3} = \frac{2}{3} \) brownies.]

Group Activity
For each of the following problems, model the solutions. DO NOT work backwards from the answer. After you have modeled each problem, write down a description of the model and the process of finding the solution. For each fraction and number stated in the problem and the answer, clearly identify the “whole” that the fraction or number refers to.

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Connection with Multiplication of Fractions

Consider the following:
• 4 groups of 5 objects = 20 objects is what we mean by $4 \times 5 = 20$
• The 5 and the 20 refer to the whole of one object: the size of the group is 5 objects and resulting total is 20 objects.
• The 4 is not 4 objects but 4 groups of objects. The whole for the 4 is different.

Consider the following:
• $\frac{1}{2}$ of a pizza = $\frac{1}{2}$ pizza
• Here the “object” (original whole) is one pizza.
• The “group” is represented by $\frac{1}{2}$ pizza.
• The $\frac{1}{2}$ refers to the group (new whole), not the original whole of one pizza.
• We could rephrase this as: Our original “object” is one pizza. We have $\frac{1}{2}$ of a group where the size of the group is $\frac{1}{2}$ pizza. What is the resulting total amount of the original object, i.e. fraction of the original one pizza?
• Hence $\frac{1}{2}$ of $\frac{1}{2}$ pizza = $\frac{1}{4}$ pizza can be represented by the expression $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$.

Order of Multiplication

• In $A \times B = C$, $B$ and $C$ refer to an original _________ (object) and $A$ refers to the number or fraction of groups.
• Note that $A$ has a different reference, a different whole. As previously with multiplication, in a given context, the _________ of the multiplication is important.
• When interpreting the order of the multiplication, rewrite the problem as A of B of the original whole. This gives you the proper order of $A \times B$.

Area Model for Multiplication of Fractions

Consider the problem of $\frac{2}{3} \times \frac{4}{5}$.

• This can be interpreted as finding the fraction (total amount) of a whole that is represented by $\frac{2}{3}$ of $\frac{4}{5}$ of the original whole, i.e. $\frac{2}{3}$ of a group where the size of the group is of an original whole.

Area Model for $\frac{2}{3} \times \frac{4}{5}$

1) Represent the original whole as a rectangle or square.
2) Represent the size of the group with respect to the whole: size of group is $\frac{4}{5}$ of rectangle. (Divide whole into 5 equal parts with vertical division and shade 4 of the parts.)
3) Represent $\frac{2}{3}$ of the size of the group, the shaded area. (Divide the shaded area into 3 equal parts using horizontal divisions and choose 2 of those parts.)

Reminder from before…

$A \times B = C$

# or fraction of groups x size of group = total
Area Model

The result of a multiplication is the total amount of the original whole, i.e. the fraction of the original square.

We count the number of equal size parts that are both shaded and cross-hatched and compare that with the total number of equal size parts in the original whole.

8 parts both cross-hatched and shaded.
15 total number of equal size parts.

Amount of original whole that is both cross-hatched and shaded is represented by \( \frac{8}{15} \).

ANSWER:

\( \frac{8}{15} \) of original whole, \( \frac{8}{15} \) of original whole, so, \( \frac{2}{3} \times \frac{4}{5} = \frac{8}{15} \).

Connection with Standard Algorithm for Multiplication of Fractions

Recall that the algorithm for multiplying fractions says to multiply the numerators, multiply the denominators, and then the answer is the product of numerators divided by product of denominators:

\[ \frac{a}{b} \times \frac{c}{d} = \frac{a \times c}{b \times d} \]

Group Activity

For each word problem:

1) A recipe calls for \( \frac{3}{7} \) ounce of chocolate. You decide that you want to make \( \frac{2}{3} \) of the recipe. What fraction of an ounce of chocolate do you use?

2) John ran \( \frac{5}{7} \) of the distance that Mike ran. Rachel ran \( \frac{3}{7} \) of the distance that John ran. What fraction of Mike's distance did Rachel run?

3) \( \frac{1}{3} \times \frac{2}{3} \) (Solve the problem with respect to the order given. Start with an original whole and represent \( \frac{1}{3} \) of the original whole and then \( \frac{2}{3} \) of the new whole)

Modeling Rational Number Multiplication

- Repeated Addition
- Joining Equal-Sized Groups
  \[ a \times b = a \text{ groups of } b \]
- Area Model
**Forms of Multiplications**

- **Multiplying Unit Fractions**
  - For rational numbers \( \frac{1}{a} \) and \( \frac{1}{b} \), \( \frac{1}{a} \times \frac{1}{b} = \frac{1}{ab} \)

- **Multiplying an Integer by a Unit Fraction**
  - For any integer \( a \) and any unit fraction \( \frac{1}{b} \), \( a \times \frac{1}{b} = \frac{a}{b} \)

- **Multiplying Rational Numbers in Fraction Form**
  - For rational numbers \( \frac{a}{b} \) and \( \frac{c}{d} \), \( \frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd} \)

**Examples**

Perform the indicated operation.

1) \( \frac{3}{2} \times \left( -\frac{4}{5} \right) \)

2) \( \left( -\frac{1}{6} \right) \times \left( -\frac{3}{8} \right) \)

3) \( 4 \times \left( \frac{2}{7} \right) \)

4) \( \left( -\frac{3}{5} \right) \times \left( -\frac{4}{5} \right) \)

**Basic Properties for Multiplication of Rational Numbers**

- **Closure Property**
  - For rational numbers \( \frac{a}{b} \) and \( \frac{c}{d} \), \( \frac{a}{b} \times \frac{c}{d} \) is a unique rational number.

- **Identity Property**
  - A unique rational number, 1, exists such that \( \frac{a}{b} \times 1 = \frac{a}{b} \) for every rational number.

- **Zero Property**
  - For each rational number \( \frac{a}{b} \), \( 0 \times \frac{a}{b} = \frac{a}{b} \times 0 = 0 \).

**Distributive Property**

For rational numbers \( \frac{a}{b} \), \( \frac{c}{d} \), and \( \frac{e}{f} \), \( \frac{a}{b} \left( \frac{c}{d} + \frac{e}{f} \right) = \frac{ac}{bd} + \frac{ae}{bf} \)

**Multiplicative Inverse**

For every nonzero rational number, \( \frac{a}{b} \), a unique rational number \( \frac{b}{a} \) exists such that \( \frac{a}{b} \times \frac{b}{a} = \frac{ab}{ab} = 1 \)

**Dividing Fractions using Area Models**

I have 2 ¼ cups of orange juice. My doctor says I can have ¾ cups of orange juice each day with my breakfast. For how many days can I have orange juice?

- We need to find how many groups of ¾ cup fit into 2 ¼ cups.
- Represent 1 cup by a rectangle, divided into 4 parts.
- Represent 2 ¼ with 2 rectangles, 1 extra part.
- 2 ¼ cups = total
- ¼ cup = size of group
- You can visualize that 3 of the ¼ cup would fit into the 2 ¼ cup:
- Answer: 3 days
Components of the Problem

I have 2 ¼ cups of orange juice. My doctor says I can have ¾ cups of orange juice each day with my breakfast. For how many days can I have orange juice?

- **Wholes**
  - 2 ¼ cup (total): 2 ¼ refers to the ‘whole’ of 1 cup orange juice.
  - ¾ cup (size of group): ¾ refers to the ‘whole’ of 1 cup orange juice and is the size of the group.
  - 3 days (number of groups): 3 refers to the ‘whole’ of 1 day that corresponds to 1 group of ¾ cup.

Group Activity

- For each of the following problems, model the solutions. After you have modeled each problem, write down a description of the model and the process of finding the solution.
- For each fraction and number stated in the problem or the answer, clearly identify the “whole” (base of reference) to which that fraction or number refers.

Group Activity

1) My mom is thinking about making a whole bunch of teddy bears. She uses ribbons for the bows. This ribbon is 5 feet long. She wants to use 1½ feet of ribbon to make a bow for each teddy bear. How many full bows can she make? How much of a bow does she have left over?

2) Barb had ¾ of a pizza left over from her party. She wants to store it in plastic containers. Each container holds ⅔ of a pizza. How many containers will she use? How many will be completely full? How full will the last container be?

3) A cookie recipe calls for ⅞ of a cup of flour. You have ⅝ cup of flour. How many recipes (or how much of a recipe) can you make?

Procedures for Dividing Fractions

- **Multiplying by the Reciprocal Method**
  - The following are used to validate multiplying by the reciprocal.
  - Common Denominator Method
  - Complex Fractions Method
  - Missing Factor Method

- **Procedures for Dividing Fractions**
  - Multiplying by the Reciprocal Method
  - When students who are familiar with the algorithm for ________ fractions are asked what the procedure is, there are a few standard answers that they tend to give.
    1. “Keep – change – change”
    2. “Change to multiplication and flip the second fraction”
    3. “Multiply by the reciprocal”

- The last answer is certainly the more sophisticated of the three, but how many students really understand the mathematics behind this rule that they take for granted? There is a funny quote from mathematicians of the 1800s regarding how to divide fractions. When asked how to divide with fractions, it is said, they often told their students, “Thine is not to reason why – just invert and multiply!” Unfortunately, most students would prefer to live by this mantra, and not ever care why this works.
Below are three methods that can be used to validate multiplying by the reciprocal.

An example of dividing fractions by multiplying by the reciprocal, as well as an example of each method is provided. Carefully compare one method to the others and see how each can be used to justify the rule: To divide fractions, multiply by the reciprocal.

Example: Multiply by the Reciprocal
\[
\frac{2}{8} \div \frac{1}{4} = \frac{3}{4}
\]

Example: Common Denominator
\[
\frac{21}{8} \div \frac{3}{4} = \frac{3}{4}
\]

Example: Complex Fractions
\[
\frac{17}{8} \div \frac{3}{4} = \frac{3}{4}
\]

Example: Missing Factor
\[
\frac{3}{4} = f, \text{ where } f \cdot \frac{3}{4} = \frac{17}{8}\]