Rational Functions

Objectives
• Find domain of rational functions.
• Use arrow notation.
• Identify vertical asymptotes.
• Identify horizontal asymptotes.
• Use transformations to graph rational functions.
• Graph rational functions.
• Identify slant (oblique) asymptotes.
• Solve applied problems with rational functions.

Definition
• rational function – a \( \frac{p(x)}{q(x)} \) of two polynomials, where \( p(x) \) and \( q(x) \) are polynomials and where \( q(x) \) is not the zero polynomial.

REVIEW
Find the domain of \( f(x) = \frac{1}{x + 4} \)

Solution:
When the denominator \( x + 4 = 0 \), we have \( x = -4 \), so the only input that results in a denominator of 0 is \( -4 \). Thus the domain is \( \{x | x \neq -4\} \) or \( (-\infty, -4) \cup (-4, \infty) \).

REVIEW: The graph of the function is the graph of \( y = \frac{1}{x} \) translated to the left 4 units.

Example
Find the domain of each rational function.

a) \( \frac{x^2 - 25}{x - 5} \)  b) \( \frac{x}{x^2 - 25} \)  c) \( \frac{x + 5}{x^2 + 25} \)

Rational Functions
• different from other functions because they have __________
• Asymptote -- a line that the graph of a function gets closer and closer to as one travels along that line in either direction
Types of Asymptotes

**Vertical Asymptote** -- occurs where the function is__________ (denominator is equal to zero)

Form: $x = a$
where $a$ is the zero of the denominator

Note: The graph__________crosses a vertical asymptote.

**Vertical asymptotes**

- Look for domain restrictions. If there are values of $x$ which result in a zero denominator, these values would create EITHER a hole in the graph or a vertical asymptote. Which?
- If the factor that creates a zero denominator cancels with a factor in the numerator, there is a hole. If you cannot cancel the factor from the denominator, a vertical asymptote exists.
- If you evaluate $f(x)$ at values that get very, very close to the $x$-value that creates a zero denominator, you notice $f(x)$ gets very, very, very large! (approaching pos. or neg. infinity as you get closer and closer to $x$)

Try this.

$f(x) = \frac{2x - 3}{x^2 - 4}$

- Determine the vertical asymptotes of the function.
- Factor to find the zeros of the denominator:
  
  $x^2 - 4 = (x + 2)(x - 2)$

- Thus the vertical asymptotes are the lines $x = -2$ and $x = 2$.

**Horizontal Asymptotes**

**Horizontal Asymptote** -- determined by the__________of the numerator and denominator

Form: $y = a$

Note: Graph can cross this asymptote.

**Example**

Find the vertical asymptotes, if any, of each rational function.

a) $f(x) = \frac{x}{x - 1}$$

b) $g(x) = \frac{x - 1}{x^2 - 1}$$

c) $h(x) = \frac{x - 1}{x^2 + 1}$

**HORIZONTAL ASYMPTOTES**

Look at the rational function $r(x) = \frac{P(x)}{Q(x)}$

- If degree of $P(x) <$ degree of $Q(x)$, horizontal asymptote $y = 0$ (x-axis).
- If degree of $P(x) =$ degree of $Q(x)$, horizontal asymptote $y = \frac{\text{leading coeff } P(x)}{\text{leading coeff } Q(x)} = \frac{a}{b}$
- If degree of $P(x) >$ degree of $Q(x)$, no horizontal asymptote
Example
Find the horizontal asymptote, if any, of each rational function.

a) \( f(x) = \frac{9x^2}{3x^2 + 1} \)

b) \( g(x) = \frac{9x}{3x^2 + 1} \)

c) \( h(x) = \frac{9x^3}{3x^2 + 1} \)

More Practice
Find the horizontal asymptote:

\[ f(x) = \frac{6x^4 - 3x^2 + 1}{9x^4 + 3x - 2} \]

Remember...
- The graph of a rational function never crosses a vertical asymptote.
- The graph of a rational function might cross a horizontal asymptote, but does not necessarily do so.

Example
Use the graph of \( f(x) = \frac{1}{x} \) to graph \( g(x) = \frac{1}{x+2} - 1 \).

Math Pick Up Lines
- “If I were a function, you would be my asymptote - I always tend towards you.”
- “Hey...nice asymptote.”

Strategy for Graphing a Rational Function

1) Find all x and y __________.
2) Find any vertical asymptotes by solving \( q(x) = 0 \).
3) Find any horizontal asymptote.
4) Plot at least one point __________ and __________ each x-intercept and vertical asymptote.
5) Graph the function between and beyond the vertical asymptotes.
Example

Graph \( f(x) = \frac{3x}{x-2} \).

Example

Graph \( f(x) = \frac{2x^2}{x^2-9} \).

Example

Graph \( f(x) = \frac{x^4}{x^2+2} \).

ANOTHER Type of ASYMPTOTE: Oblique (or Slant) Asymptote

If the degree of \( P(x) \) is one greater than the degree of \( Q(x) \), you have a slant asymptote.

To find oblique asymptote:

• Divide numerator by denominator
• Disregard remainder
• Set quotient equal to \( y \) (This gives the equation of the asymptote.)

What is the equation of the oblique asymptote?

\[
f(x) = \frac{4x^2 - 3x + 2}{2x+1}
\]

a) \( y = 4x - 3 \)
b) \( y = 2x - 5/2 \)
c) \( y = 2x - \frac{1}{2} \)
d) \( y = 4x + 1 \)

Example

Find the slant asymptote for

\[
f(x) = \frac{2x^2 - 5x + 7}{x-2}
\]
ASYMPTOTE SUMMARY
Occurrence of Lines as Asymptotes
For a rational function \( f(x) = \frac{p(x)}{q(x)} \), where \( p(x) \) and \( q(x) \)
have no common factors other than constants:

- **Vertical asymptotes** occur at any \( x \)-values that make the
denominator 0.
- **The \( x \)-axis is the horizontal asymptote** when the degree of
the numerator is less than the degree of the
denominator.
- **A horizontal asymptote other than the \( x \)-axis** occurs
when the numerator and the denominator have the same
degree.
- **A slant (or oblique) asymptote** occurs when the degree
of the numerator is one more than the degree of the
numerator. Find it by dividing.

Graph Example continued
3) Find the zeros of the numerator.
4.) We find \( f(0) \):

Thus \((0, -1)\) is the \( y \)-intercept.

Example continued
5. We find other function
values to determine the
general shape of the
graph and then draw
the graph.

Graph the following functions.

a) \( f(x) = \frac{x - 3}{x + 2} \)

b) \( f(x) = \frac{8 - x^2}{x^2 - 9} \)

Graph a:
- Vertical Asymptote
  \( x = -2 \)
- Horizontal Asymptote
  \( y = 1 \)
- \( x \)-intercept
  \((3, 0)\)
- \( y \)-intercept
  \((0, -3/2)\)
Graph b: \[ f(x) = \frac{8 - x^2}{x^2 - 9} \]

- **Vertical Asymptote**
  \( x = -3, x = 3 \)
- **Horizontal Asymptote**
  \( y = -1 \)
- **x-intercepts**
  \( (\pm 2.828, 0) \)
- **y-intercept**
  \( (0, -8/9) \)