MTH 125
3.7 Related Rates

Objectives

- Find a related rate.
- Use related rates to solve real-life problems.

Finding Related Rates

We have seen how the Chain Rule can be used to find $dy/dx$ implicitly.

Another important use of the Chain Rule is to find the rates of change of two or more related variables that are changing with respect to time.
Finding Related Rates

For example, when water is drained out of a conical tank (see Figure 3.37), the volume $V$, the radius $r$, and the height $h$ of the water level are all functions of time $t$.

Knowing that these variables are related by the equation

$$V = \frac{\pi}{3} r^2 h$$

you can differentiate implicitly with respect to $t$ to obtain the related-rate equation

$$\frac{d}{dt}(V) = \frac{d}{dt}\left(\frac{\pi}{3} r^2 h\right)$$

$$\frac{dV}{dt} = \frac{\pi}{3} \left[ r^2 \frac{dh}{dt} + h \left(2r \frac{dr}{dt}\right) \right]$$

$$= \frac{\pi}{3} \left( r^2 \frac{dh}{dt} + 2rh \frac{dr}{dt} \right).$$

From this equation you can see that the rate of change of $V$ is related to the rates of change of both $h$ and $r$. 
Example 1 – Two Rates That Are Related

Suppose $x$ and $y$ are both differentiable functions of $t$ and are related by the equation $y = x^2 + 3$. Find $dy/dt$ when $x = 1$, given that $dx/dt = 2$ when $x = 1$.

Solution:
Using the Chain Rule, you can differentiate both sides of the equation with respect to $t$.

$$y = x^2 + 3$$  \hspace{1cm} \text{Write Original equation}

$$\frac{d}{dt} [y] = \frac{d}{dt} [x^2 + 3]$$  \hspace{1cm} \text{Differentiate with respect to } t.

Example 1 – Solution

$$\frac{dy}{dt} = 2x \frac{dx}{dt}$$  \hspace{1cm} \text{Chain Rule}

When $x = 1$ and $dx/dt = 2$, you have

$$\frac{dy}{dt} = 2(1)(2)$$
$$= 4.$$
Example 2 – *Ripples in a Pond*

A pebble is dropped into a calm pond, causing ripples in the form of concentric circles, as shown in Figure 3.38.

The radius $r$ of the outer ripple is increasing at a constant rate of 1 foot per second. When the radius is 4 feet, at what rate is the total area $A$ of the disturbed water changing?

Example 2 – Solution

The variables $r$ and $A$ are related by $A = \pi r^2$. The rate of change of the radius $r$ is $dr/dt = 1$.

**Equation:** \[ A = \pi r^2 \]

**Given rate:** \[ \frac{dr}{dt} = 1 \]

**Find:** \[ \frac{dA}{dt} \] when \( r = 4 \)

With this information, you can proceed as in Example 1.
Example 2 – Solution

When the radius is 4 feet, the area is changing at a rate of \(8\pi\) square feet per second.

Substitute 4 for \(r\) and 1 for \(dr/dt\).

Differentiate with respect to \(t\).

Chain Rule

Substitute 4 for \(r\) and 1 for \(dr/dt\).

Strategy for Solving Related Rate Problems

1. Draw and label a picture. Identify all given quantities and quantities to be determined. Note what changes and what does not change.
2. Write down what is given.
3. Write down what is to be found.
4. Write an equation involving the variables whose rates of change either are given or are to be determined.
5. Using the Chain Rule, implicitly differentiate both sides of the equation with respect to time.
6. After completing step 5, substitute into the resulting equation all known values for the variables and their rates of change. Then solve for the required rate of change.
Problem Solving with Related Rates

The table below lists examples of mathematical models involving rates of change. For instance, the rate of change in the first example is the velocity of a car.

<table>
<thead>
<tr>
<th>Verbal Statement</th>
<th>Mathematical Model</th>
</tr>
</thead>
<tbody>
<tr>
<td>The velocity of a car after traveling for 1 hour is 50 miles per hour.</td>
<td>$x = \text{distance traveled}$</td>
</tr>
<tr>
<td></td>
<td>$\frac{dx}{dt} = 50$ when $t = 1$</td>
</tr>
<tr>
<td>Water is being pumped into a swimming pool at a rate of 10 cubic meters per hour.</td>
<td>$V = \text{volume of water in pool}$</td>
</tr>
<tr>
<td></td>
<td>$\frac{dV}{dt} = 10 \text{ m}^3/\text{hr}$</td>
</tr>
<tr>
<td>A gear is revolving at a rate of 25 revolutions per minute (1 revolution = $2\pi$ radians).</td>
<td>$\theta = \text{angle of revolution}$</td>
</tr>
<tr>
<td></td>
<td>$\frac{d\theta}{dt} = 25(2\pi) \text{ rad/min}$</td>
</tr>
</tbody>
</table>

EXAMPLE 3:

A child throws a stone into a still millpond causing a circular ripple to spread. If the radius increases at the constant rate of 0.5 meter per second, how fast is the area of the ripple increasing when the radius of the ripple is 20 meters?

How fast is the area changing when the radius is 20 meters?

Note: rates of change are derivatives.

rate of change of radius: $\frac{dr}{dt} = 0.5$
rate of change of area: $\frac{dA}{dt}$ when $r = 20$
We need the formula that connects the radius of the circle with its area. \[ A = \pi r^2 \]

Next, we differentiate both sides of the equation with respect to \( t \):

\[
\frac{dA}{dt} = 2\pi r \frac{dr}{dt}
\]

note: \( r' = \frac{dr}{dt} \)

\[
\frac{dA}{dt} = \pi \cdot 2r r'
\]

\[ \Rightarrow \quad \pi(2)(20)(0.5) = 20\pi \text{ m}^2/\text{s} \quad \text{or} \quad 62.8 \text{ m}^2/\text{s} \]

---

**EXAMPLE 4:**

Two cars start moving from the same point. One travels south at 60mi/h and the other travels west at 25mi/h. At what rate is the distance between the cars increasing two hours later?

Given: \( \frac{dy}{dt} = 60 \) and \( \frac{dx}{dt} = 25 \)

Find: \( \frac{dz}{dt} \) when \( t = 2h \)
What equation relates the three sides of a right triangle? \( x^2 + y^2 = z^2 \)

\[
\frac{d}{dt}(x^2) + \frac{d}{dt}(y^2) = \frac{d}{dt}(z^2) \quad \rightarrow \quad 2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 2z \frac{dz}{dt}
\]

After 2 hours: \( x = 50 \) and \( y = 120 \).

\[
z = \sqrt{50^2 + 120^2} = 130
\]

Subst gives: \( 2(50)(25) + 2(120)(60) = 2(130)(\frac{dz}{dt}) \)

After 2 hours \( \frac{dz}{dt} = \frac{2500 + 14400}{2(130)} = 65 \text{m/h} \)

EXAMPLE 5

- A ladder 10 ft long rests against a vertical wall. If the bottom of the ladder slides away from the wall at a rate of 1 ft/s, how fast is the top of the ladder sliding down the wall when the bottom of the ladder is 6 ft from the wall?

Given: \( \frac{dx}{dt} = 1 \text{ ft/s} \)

Find: \( \frac{dy}{dt} \) when \( x = 6 \) ft

\[
x^2 + y^2 = 10^2
\]

\[
\frac{d}{dt} x^2 + \frac{d}{dt} y^2 = \frac{d}{dt} z^2
\]

\[
2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 0 \quad \text{eqa 1}
\]
When \( x = 6 \):
\[
6^2 + y^2 = 100 \quad \Rightarrow \quad y = \sqrt{100 - 36} = 8
\]

Substitute into eqa 1:
\[
(2)(6)(1) + 2(8)(\frac{dy}{dt}) = 0
\]
\[
16(dy/dt) = -12 \quad \Rightarrow \quad dy/dt = -\frac{3}{4} \text{ ft/s}
\]

Example 6:
The altitude of a triangle is increasing at a rate of 1 cm/min while the area of the triangle is increasing at a rate of 2 cm²/min. At what rate is the base of the triangle changing when the altitude is 10 cm and the area is 100 cm²?

Given:
\[
\frac{dh}{dt} = 1 \text{ cm/min}
\]
\[
\frac{dA}{dt} = 2 \text{ cm}^2/\text{min}
\]

Find: \( \frac{db}{dt} \) when \( h = 10 \text{ cm} \) and \( A = 100 \text{ cm}^2 \)

\[
A = \frac{1}{2}bh
\]
\[
100 = \frac{1}{2}b(10)
\]
\[
b = 20
\]
\[
\frac{dA}{dt} = \frac{d}{dt} \left( \frac{1}{2} bh \right) = \frac{1}{2} \frac{d}{dt} (bh) = \frac{1}{2} \left( \frac{db}{dt} \cdot h + b \frac{dh}{dt} \right)
\]

\[2 = \frac{1}{2} \left( \frac{db}{dt} \cdot 10 + 20 \cdot 1 \right) \Rightarrow 2 = 5 \frac{db}{dt} + 10 \Rightarrow -8 = 5 \frac{db}{dt}
\]

\[\Rightarrow -1.6 \text{ cm/min} = \frac{db}{dt}
\]

**Ex 7:** A water tank in the shape of an inverted circular cone has a base radius of 5 ft and height 10 ft. If water is pumped into the tank at a rate of 9 ft\(^3\)/min, find the rate at which the water level is rising when the water is 6 ft deep.

Given: \(dV/dt = 9 \text{ ft}^3/\text{min}\)

Find: \(dh/dt\) when \(h = 6 \text{ ft}\)

\[V = \frac{1}{3} \pi r^2 h \quad \text{eqa 1}
\]

To find a relationship between \(r\) and \(h\), use similar triangles:

\[
\frac{r}{5} = \frac{h}{10} \quad \Rightarrow \quad r = \frac{h}{2}
\]
\[
\frac{dV}{dt} = \frac{d}{dt} \left( \frac{1}{3} \pi r^2 h \right) = \frac{d}{dt} \left( \frac{1}{3} \pi \left( \frac{h}{2} \right)^2 h \right) = \frac{d}{dt} \left( \frac{1}{3} \pi \frac{h^2}{4} h \right)
\]

\[
\frac{dV}{dt} = \frac{\pi}{12} \frac{d}{dt} (h^3) \rightarrow \frac{dV}{dt} = \frac{\pi}{12} \left( 3h^2 \frac{dh}{dt} \right)
\]

Substituting gives:

\[
9 = \frac{\pi}{12} \left( 3 \cdot 6^2 \frac{dh}{dt} \right)
\]

\[
9 = \frac{108\pi}{12} \frac{dh}{dt} \rightarrow \frac{9(12)}{108\pi} = \frac{dh}{dt}
\]

\[
\frac{dh}{dt} = \frac{1}{\pi} \approx 0.32 \text{ ft/min}
\]

Example:
Sand falls from a overhead bin, accumulating in a conical pile with a radius that is always three times its height. If the sand falls from the bin at a rate of 12 ft\(^3\)/min, how fast is the height of the sand pile changing when the pile is 10 ft high?
Example:
A spherical balloon is inflated and its volume increases at a rate of 15 in$^3$/min. What is the rate of change of its radius when the radius is 10 in?

Example:
An inverted conical water tank with a height of 12 ft and a radius of 6 ft is drained through a hole in the vertex at a rate of 2 ft$^3$/s. What is the rate of change of the water depth when the water depth is 3 ft?
An observer stands 200 m from the launch site of a hot-air balloon. The balloon rises vertically at a constant rate of 4 m/s. How fast is the angle of elevation of the balloon increasing 30 s after the launch?

As the balloon rises, its distance from the ground $y$ and its angle of elevation $\theta$ change simultaneously.

A ladder 25 feet long is leaning against the wall of a house (see figure). The base of the ladder is pulled away from the wall at a rate of 2 feet per second.

(a) How fast is the top of the ladder moving down the wall when its base is 7 feet, 15 feet, and 24 feet from the wall?
(b) Consider the triangle formed by the side of the house, the ladder, and the ground. Find the rate at which the area of the triangle is changing when the base of the ladder is 7 feet from the wall.

(c) Find the rate at which the angle between the ladder and the wall of the house is changing when the base of the ladder is 7 feet from the wall.