Inverse Functions: One-to-One Functions and Finding and Graphing the Equation of an Inverse

Objectives
- Verify inverse functions
- Use the horizontal line test to determine if a function is a one-to-one function.
- Find the inverse of a function.
- Find the inverse of a function & graph both functions simultaneously.

What is an inverse function?
- A function that “undoes” the original function.
- A function “wraps an x” and the inverse would “unwrap the x” resulting in x when the 2 functions are composed on each other.

\[ f(f^{-1}(x)) = f^{-1}(f(x)) = x \]

Example
Given that \( f(x) = 7x - 2 \) and \( g(x) = \frac{x + 2}{7} \), use composition of functions to prove or disprove that they are inverses.

Do all functions have inverses?
- Yes, and no.
  - Yes, they all will have inverses, BUT we are only interested in the inverses if they ARE A FUNCTION.

- DO ALL FUNCTIONS HAVE INVERSES THAT ARE FUNCTIONS?
  - NO.

- Recall, functions must pass the vertical line test when graphed. If the inverse is to pass the vertical line test, the original function must pass the HORIZONTAL line test (be one-to-one)!

One-to-One Functions
A function \( f(x) \) is a one-to-one function if \( x \)-values do not share the same \( y \)-values.

Remember that a function will have different \( x \)-values.

A one-to-one function will have different \( x \)-values and different \( y \)-values.
Why are one-to-one functions important?

One-to-One Functions have Inverse functions

How can you tell if a function is one-to-one?

• Use the **Horizontal Line Test** to determine whether a function is one-to-one.
  
• A function is one-to-one if and only if no horizontal line intersects its graph more than once.

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Horizontal-Line Test

Graph \( f(x) = -3x + 4 \).

**Example:** From the graph at the left, determine whether the function is one-to-one and thus has an inverse that is a function.

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Horizontal-Line Test

Graph \( f(x) = x^2 - 2 \).

**Example:** From the graph at the left, determine whether the function is one-to-one and thus has an inverse that is a function.

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How do you find an inverse?

• “Undo” the function.
  
  – Reverse the operations and order with which they occur. Great way to check the algebraic method you are REQUIRED to perform (below).

  OR

• Algebraically replace the x with y and solve for y.
  
  – This is the method you will be required to perform on tests. Steps on next slide.

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How to find the Inverse of a One-to-One Function

1. Replace \( f(x) \) with \( y \) in the equation.

2. Interchange \( x \) and \( y \) in the equation.

3. Solve this equation for \( y \).

4. Replace \( y \) with \( f^{-1}(x) \).

Any restrictions on \( x \) or \( y \) should be considered and included with the equation.

**Important:** Since everything about the inverse functions are reversed, the domain and range are interchanged for inverses, also.
Example

Determine whether the function \( f(x) = 3x^3 - 2 \) is one-to-one, and if it is, find a formula for \( f^{-1}(x) \).

How do the graphs of inverse functions compare?

- The graph of a function and its inverse always mirror each other through the line \( y = x \).
- Example: \( y = (1/3)x + 2 \) and its inverse \( = 3(x-2) \)
- Every point on the graph \((x,y)\) exists on the inverse as \((y,x)\)
  - (i.e. if \((-6,0)\) is on the graph, \((0,-6)\) is on its inverse.

Graph of Inverse \( f^{-1} \) function

- The graph of \( f^{-1} \) is obtained by reflecting the graph of \( f \) across the line \( y = x \).
- To graph the inverse \( f^{-1} \) function:
  Interchange the points on the graph of \( f \) to obtain the points on the graph of \( f^{-1} \).

Example

Graph \( f(x) = 3x - 2 \) and \( f^{-1}(x) = \frac{x + 2}{3} \) using the same set of axes.
Then compare the two graphs.

Determine the domain and range of the function and its inverse.

Solution

<table>
<thead>
<tr>
<th>( x )</th>
<th>( 3x^3 - 2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1</td>
<td>-5</td>
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<tr>
<td>0</td>
<td>-2</td>
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<tr>
<td>2</td>
<td>4</td>
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<tr>
<td>3</td>
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Properties of One-to-One Functions and Inverses

- If a function is one-to-one, then its inverse is a function.
- The domain of a one-to-one function \( f \) is the range of the inverse \( f^{-1} \).
- The range of a one-to-one function \( f \) is the domain of the inverse \( f^{-1} \).
- A function that is increasing over its domain or is decreasing over its domain is a one-to-one function.
Restricting a Domain

- When the inverse of a function is not a function, the domain of the function can be restricted to allow the inverse to be a function.
- In such cases, it is convenient to consider "part" of the function by restricting the domain of \( f(x) \). If the domain is restricted, then its inverse is a function.

Restricting the Domain

Recall that if a function is not one-to-one, then its inverse will not be a function.

Example

- For \( f(x) = x^2 - 1 \), \( x \leq 0 \):
  a.) Find the equation for the inverse, \( f^{-1} \)

b.) Find the domain and range for the function and its inverse.

If we restrict the domain values of \( f(x) \) to those greater than or equal to zero, we see that \( f(x) \) is now one-to-one and its inverse is now a function.