Markov Processes

Regular Stochastic Matrices

A stochastic matrix is said to be regular if some power has all positive entries.

Example: Regular Stochastic Matrix

Which of the following stochastic matrices are regular?

a) \[
\begin{bmatrix}
.6 & .2 \\
.4 & .8 
\end{bmatrix}
\]
b) \[
\begin{bmatrix}
0 & .5 \\
1 & .5 
\end{bmatrix}
\]
c) \[
\begin{bmatrix}
0 & 1 \\
1 & 0 
\end{bmatrix}
\]

Stable Matrix and Distribution

If a stochastic matrix \( A \) has the properties that

1. as \( n \) gets large, \( A^n \) approaches a fixed matrix, and
2. any initial distribution approaches a fixed distribution for large \( n \), then

the fixed matrix is called the stable matrix of \( A \) and the fixed distribution is called the stable distribution of \( A \).

Example: Stable Matrix and Distribution

In Jordan, 25% of the women currently work. The effect of maternal influence of mothers on their daughters is given by the matrix

\[
A = \begin{bmatrix}
.6 & .2 \\
.4 & .8 
\end{bmatrix}
\]

Find the stable matrix and the stable distribution of \( A \).

Example: Stable Matrix and Distribution (2)

\[
A^2 = \begin{bmatrix}
.44 & .56 \\
.32 & .68 
\end{bmatrix},
A^3 = \begin{bmatrix}
.32 & .68 \\
.32 & .68 
\end{bmatrix}
\]

It appears that the powers are approaching

\[
\begin{bmatrix}
\frac{1}{3} & \frac{1}{3} \\
\frac{2}{3} & \frac{2}{3} 
\end{bmatrix}
\]

which is the stable matrix.
Example: Stable Matrix and Distribution (3)

- For the initial distribution, \( \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{25}{2} \\ \frac{1}{2} & \frac{1}{2} & 25 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ \frac{2}{3} \end{bmatrix} \)

- However, for any initial distribution, \( \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & p \\ \frac{1}{2} & \frac{1}{2} & 1-p \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ \frac{2}{3} \end{bmatrix} \)
  which is the stable distribution.

Properties of Regular Stochastic Matrix

- Let \( A \) be a regular stochastic matrix.
  1. The powers \( A^n \) approach a certain matrix as \( n \) gets large. This limiting matrix is called the stable matrix of \( A \).
  2. For any initial distribution \( A^n x_0 \) approaches a certain distribution. This limiting distribution is called the stable distribution of \( A \).
  3. All columns of the stable matrix are the same; they equal the stable distribution.
  4. The stable distribution \( X = \ldots \) can be determined by solving the system of linear equations

\[
\begin{align*}
\text{sum of the entries of } X &= 1 \\
AX &= X.
\end{align*}
\]

Example: Properties Regular Matrices

- Use the properties of a regular stochastic matrix to find the stable matrix and stable distribution of \( A = \begin{bmatrix} 0.7 & 0.2 \\ 0.6 & 0.8 \end{bmatrix} \).

Example

A marketing analysis shows that 12% of the consumers who do not currently drink KickKola will purchase KickKola the next time they buy a cola and that 63% of the consumers who currently drink KickKola will purchase it the next time they buy a cola.

Make a long-range prediction of KickKola’s ultimate market share, assuming that current trends continue.