Markov Processes

The Transition Matrix

Some Background Information

- Mathematical models that evolve over time in a probabilistic manner are called **stochastic processes**.

- A special kind of stochastic process is a **Markov Chain**, where the outcome of an experiment depends only on the outcome of the previous experiment.

Why Study Markov Chains (Processes)?

- Markov chains are used to analyze trends and predict the future. (Weather, stock market, genetics, product success, etc.)

Example: Markov Process

A particular utility stock is very stable and, in the short run, the probability that it increases or decreases in price depends only on the result of the preceding day's trading. The price of the stock is observed at 4 P.M. each day and is recorded as "increased," "decreased," or "unchanged." The sequence of observations forms a Markov process.

States

- The experiments of a Markov process are performed at regular time intervals and have the same set of outcomes.
- These outcomes are called **states**, and the outcome of the current experiment is referred to as the **current state** of the process.
- The states are represented as column matrices.
Transition Matrix

• The transition matrix records all data about transitions from one state to the other. The form of a general transition matrix is

<table>
<thead>
<tr>
<th>Current state</th>
</tr>
</thead>
<tbody>
<tr>
<td>State 1</td>
</tr>
<tr>
<td>...</td>
</tr>
<tr>
<td>State j</td>
</tr>
<tr>
<td>...</td>
</tr>
<tr>
<td>State r</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Next state</th>
</tr>
</thead>
<tbody>
<tr>
<td>State 1</td>
</tr>
<tr>
<td>...</td>
</tr>
<tr>
<td>State i</td>
</tr>
<tr>
<td>...</td>
</tr>
<tr>
<td>State r</td>
</tr>
</tbody>
</table>

\[ \Pr(\text{next } i | \text{current } j) \]

Fitness Example

• A group of physical fitness devotees works out in the gym every day. The workouts vary from strenuous to moderate to light. When their exercise routine was recorded, the following observation was made: Of the people who work out strenuously on a particular day, 40% will work out strenuously on the next day and 60% will work out moderately. Of the people who work out moderately on a particular day, 50% will work out strenuously and 50% will work out lightly on the next day. Of the people working out lightly on a particular day, 30% will work out strenuously, 20% moderately, and 50% lightly.

• Using S, M, and L as row and column headings, construct a transition (stochastic) matrix for the above situation.

Stochastic Matrix

• A stochastic matrix is any square matrix that satisfies the following two properties:
  • 1. All entries are greater than or equal to 0;
  • 2. The sum of the entries in each column is 1.

All transition matrices are stochastic matrices.

Distribution Matrix

• The matrix that represents a particular state is called a distribution matrix.

• Whenever a Markov process applies to a group with members in \( r \) possible states, a distribution matrix for \( n \) is a column matrix whose entries give the percentages of members in each of the \( r \) states after \( n \) time periods.

• The initial distribution matrix describes “Generation Zero”.

Distribution Matrix for \( n \)

• Let \( A \) be the transition matrix for a Markov process with initial distribution matrix \( I_0 \), then the distribution matrix after \( n \) time periods is given by \( A^n \).
Fitness Example Continued (Method 1)

- Suppose that on a particular Monday 80% of the people at the gym have a strenuous workout, 10% have a moderate workout, and 10% have a light workout. What percent will have a strenuous workout on Wednesday?

Fitness Example Continued (Method 2)

- Suppose that on a particular Monday 80% of the people at the gym have a strenuous workout, 10% have a moderate workout, and 10% have a light workout. What percent will have a strenuous workout on Wednesday?

Interpretation of the Entries of $A^n$

The entry in the $i^{th}$ row and $j^{th}$ column of the matrix $A^n$ is the probability of the transition from state $j$ to state $i$ after $n$ periods.

Example

- A small town has only two dry cleaners, Quick & Clean and Northlake Cleaners. Quick & Clean’s manager hopes to increase the firm’s market share by conducting an extensive advertising campaign. After the campaign, a market research firm finds that there is a probability of .8 that a customer of Quick & Clean will bring his next batch of dirty clothes to Quick & Clean, and a .35 chance that a Northlake Cleaner customer will switch to Quick & Clean for his next batch. Find the probability that a person bringing his first batch to Northlake Cleaners will bring his fourth batch to Quick & Clean.