Section 2.6
Rational Functions
and
Their Graphs
Rational Functions

Rational Functions are quotients of polynomial functions. This means that rational functions can be expressed as

\[ f(x) = \frac{p(x)}{q(x)} \]

where \( p \) and \( q \) are polynomial functions and \( q(x) \neq 0 \). The domain of a rational function is the set of all real numbers except the \( x \)-values that make the denominator zero.
Example 1

Find the domain of the rational function.

\[ f(x) = \frac{x^2 - 16}{x - 4} \]
Example 2

Find the domain of the rational function.

\[ f(x) = \frac{x}{x^2 - 36} \]
\[ f(x) = \frac{1}{x} \]

**x decreases without bound:**

<table>
<thead>
<tr>
<th>(x)</th>
<th>1</th>
<th>-10</th>
<th>-100</th>
<th>-1000</th>
</tr>
</thead>
<tbody>
<tr>
<td>(f(x))</td>
<td>-1</td>
<td>-0.1</td>
<td>-0.01</td>
<td>-0.001</td>
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**x increases without bound:**

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Vertical Asymptote $x = 0$

$f(x) \to \infty$ as $x \to 0^+$

Horizontal Asymptote $y = 0$

$f(x) \to 0$ as $x \to -\infty$

$f(x) \to 0$ as $x \to \infty$

$f(x) = \frac{1}{x^2}$

$f(x) \to \infty$ as $x \to 0^+$
To find vertical asymptotes . . .

1. Factor numerator and denominator and simplify if possible.

2. Set what is left in the denominator equal to zero and solve.
What happens when factors cancel?

$$f(x) = \frac{x^2 - 4}{x - 2} = \frac{(x + 2)(x - 2)}{x - 2} = x + 2, \ x \neq 2.$$ 

Denominator is zero at $x = 2$.

In this reduced form, 2 does not result in a zero denominator.

A graph with a hole corresponding to the denominator’s zero. Your calculator will not show the hole.
Two Graphs With Vertical Asymptotes, One Without

**Figure 3.28(a)**
The graph of \( f(x) = \frac{x}{x^2 - 9} \) has two vertical asymptotes.

**Figure 3.28(b)**
The graph of \( g(x) = \frac{x + 3}{x^2 - 9} \) has one vertical asymptote.

**Figure 3.28(c)**
The graph of \( h(x) = \frac{x + 3}{x^2 + 9} \) has no vertical asymptotes.
Example 3

Find the vertical asymptote, if any, of the graph of the rational function.

\[ f(x) = \frac{x}{x^2 - 36} \]
Example 4

Find the vertical asymptote, if any, of the graph of the rational function.

\[ f(x) = \frac{x}{x^2 + 36} \]
Example 5

Find the vertical asymptote, if any, of the graph of the rational function.

\[ f(x) = \frac{x + 6}{x^2 - 36} \]
Definition of a Horizontal Asymptote

The line \( y = b \) is a **horizontal asymptote** of the graph of a function \( f(x) \) if \( f(x) \) approaches \( b \) as \( x \) increases or decreases without bound.

**As \( x \to \infty, f(x) \to b.**

Locating Horizontal Asymptotes

Let \( f \) be the rational function given by

\[
    f(x) = \frac{a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0}{b_m x^m + b_{m-1} x^{m-1} + \cdots + b_1 x + b_0}, \quad a_n \neq 0, \ b_m \neq 0.
\]

The degree of the numerator is \( n \). The degree of the denominator is \( m \).

1. If \( n < m \), the \( x \)-axis, or \( y = 0 \), is the horizontal asymptote of the graph of \( f \).
2. If \( n = m \), the line \( y = \frac{a_n}{b_m} \) is the horizontal asymptote of the graph of \( f \).
3. If \( n > m \), the graph of \( f \) has no horizontal asymptote.
Two Graphs With Horizontal Asymptotes, One Without

**Figure 3.30(a)** The horizontal asymptote of the graph is $y = 0$.  

**Figure 3.30(b)** The horizontal asymptote of the graph is $y = 2$.  

**Figure 3.30(c)** The graph has no horizontal asymptote.
Example 6

Find the horizontal asymptote, if any, of the graph of the rational function.

\[ f(x) = \frac{3x}{x^2 + 1} \]
Example 7

Find the horizontal asymptote, if any, of the graph of the rational function.

\[ f(x) = \frac{6x^2}{x^2 + 1} \]
Example 8

Find the horizontal asymptote, if any, of the graph of the rational function.

\[ f(x) = \frac{5x^3 - 2x + 1}{x^2} \]
Transformations of Rational Functions

Begin with \( f(x) = \frac{1}{x^2} \). We've identified two points and the asymptotes.

Graph \( y = \frac{1}{(x - 2)^2} \):
Shift 2 units to the right. Add 2 to each x-coordinate.

The graph of \( y = \frac{1}{(x - 2)^2} \), showing two points and the asymptotes.

Graph \( g(x) = \frac{1}{(x - 2)^2} + 1 \),
Shift 1 unit up. Add 1 to each y-coordinate.

The graph of \( g(x) = \frac{1}{(x - 2)^2} + 1 \),
showing two points and the asymptotes.
Use the graph of $f(x) = \frac{1}{x}$ to graph $g(x) = \frac{1}{x-3} - 4$
Strategy for Graphing a Rational Function

The following strategy can be used to graph

\[ f(x) = \frac{p(x)}{q(x)}, \]

where \( p \) and \( q \) are polynomial functions with no common factors.

1. Determine whether the graph of \( f \) has symmetry.

\[ f(-x) = f(x): \text{y-axis symmetry} \]
\[ f(-x) = -f(x): \text{origin symmetry} \]

2. Find the \( y \)-intercept (if there is one) by evaluating \( f(0) \).
3. Find the \( x \)-intercepts (if there are any) by solving the equation \( p(x) = 0 \).
4. Find any vertical asymptote(s) by solving the equation \( q(x) = 0 \).
5. Find the horizontal asymptote (if there is one) using the rule for determining the horizontal asymptote of a rational function.
6. Plot at least one point between and beyond each \( x \)-intercept and vertical asymptote.
7. Use the information obtained previously to graph the function between and beyond the vertical asymptotes.
Example 10

Graph \( f(x) = \frac{2x}{x-2} \).

(Factored & simplified?)
1. Symmetric?

2. \( y \)-intercept?

3. \( x \)-intercept(s)?

4. Vertical Asymptote(s)?

5. Horizontal Asymptote(s)?

6. Extra Points:
Example 11 \[ f(x) = \frac{2x^2}{x^2 - 25}. \]  

(Factored & simplified?)  
1. Symmetric?  
2. \( y \)-intercept?  
3. \( x \)-intercept(s)?  
4. Vertical Asymptote(s)?  
5. Horizontal Asymptote(s)?  
6. Extra Points:
The graph of a rational function has a slant asymptote if the degree of the numerator is one more than the degree of denominator. The equation of the slant asymptote can be found by division. It is the equation of the dividend with the term containing the remainder dropped.

\[
\frac{p(x)}{q(x)} = mx + b + \frac{\text{remainder}}{q(x)}
\]

**Slant Asymptote:** \( y = mx + b \)
Example 12

Find the slant asymptote of the function \( f(x) = \frac{x^2 + 6x + 2}{x} \).
Example 13

Find the slant asymptote of the function \( f(x) = \frac{x^3 - 1}{x^2 + 2x + 2} \).
Concept Check  \( f(x) = \frac{3x^2 + 8x - 3}{x^2 + 3x} \)

1. \( y \)-intercept?

2. \( x \)-intercept(s)?

3. Vertical Asymptote(s)?

4. Horizontal or Slant Asymptote(s)?
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