Section 2.3
Polynomial Functions and Their Graphs
What is a Polynomial Function?

**Definition of a Polynomial Function**

Let \( n \) be a nonnegative integer and let \( a_n, a_{n-1}, \ldots, a_2, a_1, a_0 \) be real numbers, with \( a_n \neq 0 \). The function defined by

\[
f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0
\]

is called a polynomial function of degree \( n \). The number \( a_n \) is called the leading coefficient.
Example 1

Determine if it’s a polynomial. If so, state the degree. If not, state why.

a. \( f(x) = 5 - 3x^{-1} + 5x^2 \)

b. \( g(x) = 28 - x^4 + x^3 \)

c. \( y = 8 \)

d. \( h(x) = \frac{1}{x+2} \)

e. \( f(x) = 20x^2 - 4x^{2/3} \)
Characteristics of Polynomials

• They are *smooth* and *continuous*. 
Examples of Functions That Aren’t Polynomials

- Polynomials *don’t have breaks or sharp turns* in their graphs.
Analyzing a Polynomial Equation

\[ f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0 \]

<table>
<thead>
<tr>
<th>( n ) odd</th>
<th>( n ) even</th>
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<tbody>
<tr>
<td>( a^n &gt; 0 )</td>
<td>Falls left, rises right</td>
</tr>
<tr>
<td>( a^n &lt; 0 )</td>
<td>Rises left, falls right</td>
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Example 2

Use the leading coefficient to determine end behavior.

a. \( f(x) = -x^2 - x^3 \)

b. \( h(x) = 4 + 7x^4 - 5x^3 + x \)

c. \( g(x) = 2(x + 5)^9 \)
Remember Our Graphs . . .

- We can use the functions to tell us a lot more about the corresponding graphs.
$x$ — intercepts

- $x$ — intercepts are the real zeros of $f(x)$ — i.e. the roots or solutions when you set $f(x) = 0$.

- So, as usual, to find the above we just set the equation equal to 0 and solve.

- A polynomial of degree $n$ can have at most $n$ real roots.
Example 3

Find all the zeros of \( f(x) = x^3 + 5x^2 - 9x - 45 \).
What does that mean in the graph?

\[ f(x) = x^3 + 5x^2 - 9x - 45 = (x - 3)(x + 3)(x + 5) \]
Example 4

Find all the zeros of $f(x) = 9 - 4x^2$. 
Example 5

Find all the zeros of \( f(x) = x^4 - 2x^2 + 1 \).
Example 6

Find all the zeros of $f(x) = -2(x + 3)^2(x - 4)^3$.

Degree?
Leading coefficient?
What does that mean in the graph?

\[ f(x) = -2x^3 - 4x^2 + 30x + 72 = -2(x - 4)(x + 3)^2 \]
Multiplicity of Roots

• The *multiplicity* of a root is the degree of the linear factor associated with that root.

• Ex:

\[ f(x) = -3(x + 1)^3(x - 2)^2(x - 3) \]
Multiplicities of Roots

- **Even Multiplicity** – graph *touches*, but does not cross, the $x$–axis.
- **Odd Multiplicity** – graph *crosses* the $x$–axis.
Example 7

Find the zeros, give the multiplicity, and tell whether it touches or crosses the $x$-axis.

$$f(x) = 11(x - 8)^4(x + 3)^5$$
Example 8

Do the same for Example 3.

\[ f(x) = x^3 + 5x^2 - 9x - 45 \]

yielded

\[ f(x) = (x - 3)(x + 3)(x + 5). \]
What did the graph show us?

\[ f(x) = x^3 + 5x^2 - 9x - 45 = (x - 3)(x + 3)(x + 5) \]
Extrema (Peaks and Valleys)

- Can also be thought of as “turning points.”
- Places where the graph switches from increasing to decreasing.
- A polynomial of degree $n$ can have at most $n-1$ “turning points.”
Remember Our Graphs . . .

- What are the lowest possible degrees of these?
The Intermediate Value Theorem

Let $f$ be a polynomial function with real coefficients. If $f(a)$ and $f(b)$ have opposite signs, then there is at least one value of $c$ between $a$ and $b$ for which $f(c) = 0$.

**Example 7**

Use the intermediate value theorem to show there exists a real zero between the given integers. Find the $c$ from above.

$$f(x) = x^3 - x; \text{ between } -2 \text{ and } 2.$$
Thinking Problem

Find a polynomial of equation of least degree that this graph could represent.
Questions???

Don’t forget to be working on the homeworks and quizzes now, not later.