Area of a Region Between Two Curves

With a few modifications, you can extend the application of definite integrals from the area of a region under a curve to the area of a region between two curves.

Consider two functions $f$ and $g$ that are continuous on the interval $[a, b]$. 

Figure 7.1
Area of a Region Between Two Curves

If, as in Figure 7.1, the graphs of both $f$ and $g$ lie above the $x$-axis, and the graph of $g$ lies below the graph of $f$, you can geometrically interpret the area of the region between the graphs as the area of the region under the graph of $g$ subtracted from the area of the region under the graph of $f$, as shown in Figure 7.2.

$$\int_a^b [f(x) - g(x)] \, dx = \int_a^b f(x) \, dx - \int_a^b g(x) \, dx$$

Figure 7.2

Area of a Region Between Two Curves

To verify the reasonableness of the result shown in Figure 7.2, you can partition the interval $[a, b]$ into $n$ subintervals, each of width $\Delta x$.

Then, as shown in Figure 7.3, sketch a representative rectangle of width $\Delta x$ and height $f(x_i) - g(x_i)$, where $x_i$ is in the $i$th subinterval.

Figure 7.3
Area of a Region Between Two Curves

The area of this representative rectangle is
\[ \Delta A_i = (\text{height})(\text{width}) = [f(x_i) - g(x_i)]\Delta x. \]

By adding the areas of the \( n \) rectangles and taking the limit as \( ||\Delta|| \to 0 \) \( (n \to \infty) \), you obtain
\[
\lim_{n \to \infty} \sum_{i=1}^{n} [f(x_i) - g(x_i)]\Delta x.
\]

Because \( f \) and \( g \) are continuous on \([a, b]\), \( f - g \) is also continuous on \([a, b]\) and the limit exists. So, the area of the given region is
\[
\text{Area} = \lim_{n \to \infty} \sum_{i=1}^{n} [f(x_i) - g(x_i)]\Delta x
= \int_{a}^{b} [f(x) - g(x)]\,dx.
\]

Area of a Region Between Two Curves

If \( f \) and \( g \) are continuous on \([a, b]\) and \( g(x) \leq f(x) \) for all \( x \) in \([a, b]\), then the area of the region bounded by the graphs of \( f \) and \( g \) and the vertical lines \( x = a \) and \( x = b \) is
\[
A = \int_{a}^{b} [f(x) - g(x)]\,dx.
\]
Area of a Region Between Two Curves

In Figure 7.1, the graphs of $f$ and $g$ are shown above the $x$-axis. This, however, is not necessary.

The same integrand $[f(x) - g(x)]$ can be used as long as $f$ and $g$ are continuous and $g(x) \leq f(x)$ for all $x$ in the interval $[a, b]$.

![Figure 7.1 Region between two curves](image)

This is summarized graphically in Figure 7.4.

Notice in Figure 7.4 that the height of a representative rectangle is $f(x) - g(x)$ regardless of the relative position of the $x$-axis.

![Figure 7.4 Region between two curves](image)
Area of a Region Between Two Curves

Representative rectangles are used throughout this chapter in various applications of integration.

A vertical rectangle (of width $\Delta x$) implies integration with respect to $x$, whereas a horizontal rectangle (of width $\Delta y$) implies integration with respect to $y$.

Example 1 – Finding the Area of a Region Between Two Curves

Find the area of the region bounded by the graphs of $f(x) = x^2 + 2$, $g(x) = -x$, $x = 0$, and $x = 1$.

Solution:
Let $g(x) = -x$ and $f(x) = x^2 + 2$.

Then $g(x) \leq f(x)$ for all $x$ in $[0, 1]$, as shown in Figure 7.5.
Example 1 – Solution

So, the area of the representative rectangle is

\[ \Delta A = [f(x) - g(x)] \Delta x \]

\[ = [(x^2 + 2) - (-x)] \Delta x \]

and the area of the region is

\[ A = \int_a^b [f(x) - g(x)] \, dx = \int_0^1 [(x^2 + 2) - (-x)] \, dx \]

\[ = \left[ \frac{x^3}{3} + \frac{x^2}{2} + 2x \right]_0^1 \]

\[ = \frac{1}{3} + \frac{1}{2} + 2 \]

\[ = \frac{17}{6}. \]

Area of a Region Between Intersecting Curves
Area of a Region Between Intersecting Curves

In Example 1, the graphs of \( f(x) = x^2 + 2 \) and \( g(x) = -x \) do not intersect, and the values of \( a \) and \( b \) are given explicitly.

A more common problem involves the area of a region bounded by two *intersecting* graphs, where the values of \( a \) and \( b \) must be calculated.

Example 2 – A Region Lying Between Two Intersecting Graphs

Find the area of the region bounded by the graphs of \( f(x) = 2 - x^2 \) and \( g(x) = x \).

**Solution:**

In Figure 7.6, notice that the graphs of \( f \) and \( g \) have two points of intersection.
Example 2 – Solution

To find the $x$-coordinates of these points, set $f(x)$ and $g(x)$ equal to each other and solve for $x$.

\[
\begin{align*}
2 - x^2 &= x & \text{Set } f(x) \text{ equal to } g(x). \\
-x^2 - x + 2 &= 0 & \text{Write in general form.} \\
-(x + 2)(x - 1) &= 0 & \text{Factor.} \\
x &= -2 \text{ or } 1 & \text{Solve for } x.
\end{align*}
\]

So, $a = -2$ and $b = 1$.

Example 2 – Solution

Because $g(x) \leq f(x)$ for all $x$ in the interval $[-2, 1]$, the representative rectangle has an area of

\[
\Delta A = [f(x) - g(x)] \Delta x
\]

\[
= [(2 - x^2) - x] \Delta x
\]

and the area of the region is

\[
A = \int_{-2}^{1} [(2 - x^2) - x] \, dx = \left[ -\frac{x^3}{3} - \frac{x^2}{2} + 2x \right]_{-2}^{1} = \frac{9}{2}
\]
Set up the definite integral that gives the area of the region.

\[ y_1 = x^2 + 2x + 1 \]
\[ y_2 = 2x + 5 \]

\[ y_1 = (x - 1)^3 \]
\[ y_2 = x - 1 \]
Sketch the graph bounded by the graphs of the algebraic functions and find the area of the region.

\[ y = -x^3 + x, \quad y = x, \quad x = -1, \quad x = 1 \]

\[ f(x) = -x^2 + 4x + 1, \quad g(x) = x + 1 \]
\[ f(y) = y(2 - y), \quad g(y) = -y \]

\[ f(y) = \frac{y}{\sqrt{16 - y^2}}, \quad g(y) = 0, \quad y = 3 \]