### 5.7
The Natural Logarithmic Function: Integration

We have studied two differentiation rules for logarithms. The differentiation rule \(\frac{d}{dx}[\ln x] = \frac{1}{x}\) produces the Log Rule for Integration.

The differentiation rule \(\frac{d}{dx}[\ln u] = u'/u\) produces the integration rule
\[
\int \frac{1}{u} \, dx = \ln|u| + C.
\]

These rules are summarized below.

<table>
<thead>
<tr>
<th>THEOREM 5.21 LOG RULE FOR INTEGRATION</th>
</tr>
</thead>
<tbody>
<tr>
<td>Let (u) be a differentiable function of (x).</td>
</tr>
<tr>
<td>1. (\int \frac{1}{x} , dx = \ln</td>
</tr>
<tr>
<td>2. (\int \frac{1}{u} , du = \ln</td>
</tr>
</tbody>
</table>

Because \(du = u' \, dx\), the second formula can also be written as

\[
\int \frac{u'}{u} \, dx = \ln|u| + C.
\]

**Alternative form of Log Rule**
Log Rule for Integration

With antiderivatives involving logarithms, it is easy to obtain forms that look quite different but are still equivalent.

For instance, both of the following are equivalent to the antiderivative.

\[
\ln|3x + 2|^{1/3} + C \quad \text{and} \quad \ln|3x + 2|^{1/3} + C
\]

Integrals to which the Log Rule can be applied often appear in disguised form. For instance, if a rational function has a numerator of degree greater than or equal to that of the denominator, division may reveal a form to which you can apply the Log Rule.

Log Rule for Integration

To master integration technique, you must recognize the “form-fitting” nature of integration. In this sense, integration is not nearly as straightforward as differentiation.

Differentiation takes the form

“Here is the question; what is the answer?”

Integration is more like

“Here is the answer; what is the question?”
Log Rule for Integration

The following are guidelines you can use for integration.

GUIDELINES FOR INTEGRATION
1. Learn a basic list of integration formulas. (By the end of Section 5.8, this list will have expanded to 20 basic rules.)
2. Find an integration formula that resembles all or part of the integrand, and, by trial and error, find a choice of $u$ that will make the integrand conform to the formula.
3. If you cannot find a $u$-substitution that works, try altering the integrand. You might try a trigonometric identity, multiplication and division by the same quantity, addition and subtraction of the same quantity, or long division. Be creative.
4. If you have access to computer software that will find antiderivatives symbolically, use it.

Find the indefinite integral.

$$\int \frac{1}{x-5} \, dx$$

$$\int \frac{x^2}{5-x^3} \, dx$$
\[
\int \frac{2x^2 + 7x - 3}{x-2} \, dx
\]

\[
\int \frac{x^2 - 3x^2 + 4x - 9}{x^2 + 3} \, dx
\]

\[
\int \frac{1}{x \ln x} \, dx
\]

\[
\int \frac{1}{x^{2/3} (1 + x^{2/3})} \, dx
\]
\[
\int \frac{x(x-2)}{(x-1)^3} \, dx
\]

Find the indefinite integral by \( u \)-substitution. (Hint: Let \( u \) be the denominator of the integral.

\[
\int \frac{1}{1+\sqrt{3x}} \, dx
\]

\[
\int \frac{\sqrt{x}}{\sqrt{x} - 1} \, dx
\]
Integrals of Trigonometric Functions

We have looked at six trigonometric integration rules—the six that correspond directly to differentiation rules. With the Log Rule, you can now complete the set of basic trigonometric integration formulas.

Using a Trigonometric Identity

Find \( \int \tan x \, dx \).

Solution:
This integral does not seem to fit any formulas on our basic list.

However, by using a trigonometric identity, you obtain the following.

\[
\int \tan x \, dx = \int \frac{\sin x}{\cos x} \, dx.
\]
Solution

Knowing that $D_x[\cos x] = -\sin x$, you can let $u = \cos x$ and write

$$\int \tan x \, dx = -\int \frac{-\sin x}{\cos x} \, dx$$

Trigonometric identity

$$= -\int \frac{u'}{u} \, dx$$

Substitute: $u = \cos x$.

$$= -\ln|u| + C$$

Apply Log Rule.

$$= -\ln|\cos x| + C.$$  

Back-substitute.

Integrals of Trigonometric Functions

Example 8 uses a trigonometric identity to derive an integration rule for the tangent function.

All six trigonometric rules are summarized below.

<table>
<thead>
<tr>
<th>INTEGRALS OF THE SIX BASIC TRIGONOMETRIC FUNCTIONS</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\int \sin u , du = -\cos u + C$</td>
</tr>
<tr>
<td>$\int \tan u , du = -\ln</td>
</tr>
<tr>
<td>$\int \sec u , du = \ln</td>
</tr>
</tbody>
</table>
Find the indefinite integral.

\[ \int \tan 5\theta \, d\theta \]

\[ \int \sec \frac{x}{2} \, dx \]

\[ \int \left( 2 - \tan \frac{\theta}{4} \right) \, d\theta \]

\[ \int \frac{\csc^2 t}{\cot t} \, dt \]
\[ \int (\sec 2x + \tan 2x)dx \]

\[ \int \sec t(\sec t + \tan t)dt \]

Evaluate the definite integral.

\[ \int_{1}^{2} \frac{1}{2x + 3}dx \]

\[ \int_{e}^{2} \frac{1}{x \ln x}dx \]
\[
\int_{0.1}^{0.2} (\csc 2\theta - \cot 2\theta)^2 \, d\theta
\]