3.6 Derivatives of Inverse Trigonometric Functions

Recall that the derivative of the transcendental function $f(x) = \ln x$ is the algebraic function $f'(x) = 1/x$.

You will now see that the derivatives of the inverse trigonometric functions also are algebraic (even though the inverse trigonometric functions are themselves transcendental).

Derivatives of Inverse Trigonometric Functions

The following theorem lists the derivatives of the six inverse trigonometric functions.

<table>
<thead>
<tr>
<th>THEOREM 3.18 DERIVATIVES OF INVERSE TRIGONOMETRIC FUNCTIONS</th>
</tr>
</thead>
<tbody>
<tr>
<td>Let $u$ be a differentiable function of $x$.</td>
</tr>
<tr>
<td>$\frac{d}{dx}[\arcsin u] = \frac{u'}{\sqrt{1 - u^2}}$</td>
</tr>
<tr>
<td>$\frac{d}{dx}[\arccos u] = \frac{-u'}{\sqrt{1 - u^2}}$</td>
</tr>
<tr>
<td>$\frac{d}{dx}[\arctan u] = \frac{u'}{1 + u^2}$</td>
</tr>
<tr>
<td>$\frac{d}{dx}[\arccot u] = \frac{-u'}{1 + u^2}$</td>
</tr>
<tr>
<td>$\frac{d}{dx}[\text{arcsec} u] = \frac{u'}{</td>
</tr>
<tr>
<td>$\frac{d}{dx}[\text{arccsc} u] = \frac{-u'}{</td>
</tr>
</tbody>
</table>
Derivatives of Inverse Trigonometric Functions

Note that the derivatives of arccos \( u \), arccot \( u \), and arccsc \( u \) are the negatives of the derivatives of arcsin \( u \), arctan \( u \), and arcsec \( u \), respectively.

There is no common agreement on the definition of arcsec \( x \) (or arccsc \( x \)) for negative values of \( x \).

When we defined the range of the arcsecant, we chose to preserve the reciprocal identity arcsec \( x = \arccos(1/x) \).

For example, to evaluate arcsec\((-2)\), you can write

\[
\text{arcsec}(-2) = \arccos(-0.5) = 2.09.
\]

Derivatives of Inverse Trigonometric Functions

One of the consequences of the definition of the inverse secant function given in this section is that its graph has a positive slope at every \( x \)-value in its domain.

This accounts for the absolute value sign in the formula for the derivative of arcsec \( x \).
Example – Differentiating Inverse Trigonometric Functions

\[ a. \quad \frac{d}{dx}[\arcsin(2x)] = \frac{2}{\sqrt{1 - (2x)^2}} \quad u = 2x \]

\[ = \frac{2}{\sqrt{1 - 4x^2}} \]

\[ b. \quad \frac{d}{dx}[\arctan(3x)] = \frac{3}{1 + (3x)^2} \quad u = 3x \]

\[ = \frac{3}{1 + 9x^2} \]

Example – Differentiating Inverse Trigonometric Functions

\[ \text{cont'd} \]

\[ c. \quad \frac{d}{dx}[\arcsin \sqrt{x}] = \frac{(1/2)x^{-1/2}}{\sqrt{1 - x}} \quad u = \sqrt{x} \]

\[ = \frac{1}{2\sqrt{x}\sqrt{1 - x}} \]

\[ = \frac{1}{2\sqrt{x - x^2}} \]

\[ d. \quad \frac{d}{dx}[\text{arcsec } e^{2x}] = \frac{2e^{2x}}{e^{2x}\sqrt{(e^{2x})^2 - 1}} \quad u = e^{2x} \]
Example – Differentiating Inverse Trigonometric Functions

\[ \frac{d}{dx} \left[ \text{arccsc } e^{2x} \right] = \frac{2e^{2x}}{e^{2x} \sqrt{e^{4x} - 1}} \]

\[ = \frac{2e^{2x}}{e^{2x} \sqrt{e^{4x} - 1}} \]

\[ = \frac{2}{\sqrt{e^{4x} - 1}} \]

In part (d), the absolute value sign is not necessary because \( e^{2x} > 0 \).

Review of Basic Differentiation Rules

An elementary function is a function from the following list or one that can be formed as the sum, product, quotient, or composition of functions in the list.

<table>
<thead>
<tr>
<th>Algebraic Functions</th>
<th>Transcendental Functions</th>
</tr>
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<tbody>
<tr>
<td>Polynomial functions</td>
<td>Logarithmic functions</td>
</tr>
<tr>
<td>Rational functions</td>
<td>Exponential functions</td>
</tr>
<tr>
<td>Functions involving radicals</td>
<td>Trigonometric functions</td>
</tr>
<tr>
<td></td>
<td>Inverse trigonometric functions</td>
</tr>
</tbody>
</table>
Review of Basic Differentiation Rules

With the differentiation rules, we can differentiate any elementary function.

For convenience, these differentiation rules are summarized as below.

### BASIC DIFFERENTIATION RULES FOR ELEMENTARY FUNCTIONS

1. \( \frac{d}{dx}(cu) = cu' \)
2. \( \frac{d}{dx}(u \pm v) = u' \pm v' \)
3. \( \frac{d}{dx}(u^n) = nu^{n-1}u' \)
4. \( \frac{d}{dx}\left(\frac{u}{v}\right) = \frac{vu' - uv'}{v^2} \)
5. \( \frac{d}{dx}[c] = 0 \)
6. \( \frac{d}{dx}[a^x] = na^{x-1}u' \)
7. \( \frac{d}{dx}[x] = 1 \)
8. \( \frac{d}{dx}[u^n] = \frac{u'}{n^u}, \ u \neq 0 \)
9. \( \frac{d}{dx}[\ln u] = \frac{u'}{u} \)
10. \( \frac{d}{dx}[e^u] = e^u u' \)
11. \( \frac{d}{dx}[\log_a u] = \frac{u'}{u \ln a} \)
12. \( \frac{d}{dx}[a^u] = (\ln a)a^u u' \)
13. \( \frac{d}{dx}[\sin u] = (\cos u)u' \)
14. \( \frac{d}{dx}[\cos u] = -(\sin u)u' \)
15. \( \frac{d}{dx}[\tan u] = (\sec^2 u)u' \)
16. \( \frac{d}{dx}[\cot u] = -(\csc^2 u)u' \)
17. \( \frac{d}{dx}[\sec u] = (\sec u \tan u)u' \)
18. \( \frac{d}{dx}[\csc u] = -(\sec u \cot u)u' \)
19. \( \frac{d}{dx}[\arcsin u] = \frac{u'}{\sqrt{1 - u^2}} \)
20. \( \frac{d}{dx}[\arccos u] = -\frac{u'}{\sqrt{1 - u^2}} \)
21. \( \frac{d}{dx}[\arctan u] = \frac{u'}{1 + u^2} \)
22. \( \frac{d}{dx}[\arccot u] = -\frac{u'}{1 + u^2} \)
23. \( \frac{d}{dx}[\arcsec u] = \frac{u'}{|u| \sqrt{u^2 - 1}} \)
24. \( \frac{d}{dx}[\arccsc u] = -\frac{u'}{|u| \sqrt{u^2 - 1}} \)
Find an equation of the tangent line to the graph of \( f \) at the given point.

\[
f(x) = \arccsc x \quad \left( \sqrt{2}, \frac{\pi}{4} \right)
\]

Find \( \frac{dy}{dx} \) at the given point for the equation.

\[
x = 2 \ln(y^2 - 3), \quad (0, 2)
\]
Find the derivative of the function.

\[ f(t) = \arcsin t^2 \]

\[ f(x) = \text{arc sec } 4x \]
\[ h(x) = x^2 \arctan 5x \]

\[ g(x) = e^{2x} \arcsin x \]
Find an equation of the tangent line to the graph of the equation at the given point.

\[ \text{arc tan}(xy) = \text{arcsin}(x + y), \quad (0,0) \]