3.4 The Chain Rule

This section has yet to discuss one of the most powerful differentiation rules—the **Chain Rule**.

This rule deals with composite functions and adds a surprising versatility to the rules that we have discussed.

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**The Chain Rule**

For example, compare the following functions. Those on the left can be differentiated without the Chain Rule, and those on the right are best differentiated with the Chain Rule.

<table>
<thead>
<tr>
<th>Without the Chain Rule</th>
<th>With the Chain Rule</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y = x^2 + 1$</td>
<td>$y = \sqrt{x^2 + 1}$</td>
</tr>
<tr>
<td>$y = \sin x$</td>
<td>$y = \sin 6x$</td>
</tr>
<tr>
<td>$y = 3x + 2$</td>
<td>$y = (3x + 2)^5$</td>
</tr>
<tr>
<td>$y = e^x + \tan x$</td>
<td>$y = e^{5x} + \tan x^2$</td>
</tr>
</tbody>
</table>
The Chain Rule

When applying the Chain Rule, it is helpful to think of the composite function \( f \circ g \) as having two parts—an inner part and an outer part.

\[ y = f(g(x)) = f(u) \]

The Chain Rule

The derivative of \( y = f(u) \) is the derivative of the outer function (at the inner function \( u \)) \textit{times} the derivative of the inner function.

\[ y' = f'(u) \cdot u' \]
OUTSIDE-INSIDE” RULE

Note: we differentiate the outer function \( f \) [at the inner function \( g(x) \)] and then multiply by the derivative of the inner function.

Ex. Differentiate: \( F(x) = \sqrt{x^2 + 1} \)

\[
F(x) = (x^2 + 1)^{1/2}
\]

- What is the outer function? square root function
- What is the inside function? \( x^2 + 1 \)

\[
F'(x) = \frac{1}{2} (x^2 + 1)^{-\frac{1}{2}} \cdot (2x)
\]

\[
F'(x) = \frac{x}{\sqrt{x^2 + 1}}
\]
Transcendental Functions and the Chain Rule

The “Chain Rule versions” of the derivatives of the six trigonometric functions and the natural exponential function are as follows.

\[
\frac{d}{dx} \sin u = (\cos u) \ u' \\
\frac{d}{dx} \cos u = -(\sin u) \ u' \\
\frac{d}{dx} \tan u = (\sec^2 u) \ u' \\
\frac{d}{dx} \cot u = -(\csc^2 u) \ u' \\
\frac{d}{dx} \sec u = (\sec u \tan u) \ u' \\
\frac{d}{dx} \csc u = -(\csc u \cot u) \ u' \\
\frac{d}{dx} e^u = e^u \ u'
\]
Example  Applying the Chain Rule to Transcendental Functions

\[ y = \sin 2x \]

\[ \frac{dy}{dx} = \cos 2x \cdot \frac{d}{dx}[2x] = (\cos 2x)(2) = 2 \cos 2x \]

\[ y = \cos(x - 1) \]

\[ \frac{dy}{dx} = -\sin(x - 1) \cdot \frac{d}{dx}[x - 1] = -\sin(x - 1) \]

Ex. Differentiate: \( y = \sin(4x) \)

- What is the outside function? sine function
  - \( y' = \cos(4x) \cdot \frac{d}{dx}[4x] = 4 \cos(4x) \)

- What is the inside function? \( 4x \)
  - At the inner function \( \cdot 4 \)

- Derivative of inner function \( \cdot \)
  - Derivative of outer function \( \)
Ex. Find the derivative.
\[ y = \sin(x\cos x) \]

- \[ y' = \cos(x\cos x) \cdot (\cos x - x\sin x) \]
  
  derivative of outside function  
  inside function left alone  
  derivative of inside function  
  (use product rule)

\[ y' = (\cos x - x\sin x)[\cos(x\cos x)] \]

Example: Find the derivative
\[ y = \tan^2(3\theta) \]

note: \[ \tan^2 (3\theta) = (\tan(3\theta))^2 \]

- \[ y' = 2(\tan(3\theta)) \cdot \sec^2(3\theta) \cdot 3 \]
  
  derivative of outer function  
  (inner function unchanged)  
  derivative of inner function

\[ y' = 6\tan(3\theta)\sec^2(3\theta) \]
The Derivative of the Natural Logarithmic Function

Up to this point, derivatives of algebraic functions have been algebraic and derivatives of transcendental functions have been transcendental.

The next theorem looks at an unusual situation in which the derivative of a transcendental function is algebraic. Specifically, the derivative of the natural logarithmic function is the algebraic function $1/x$.

Let $u$ be a differentiable function of $x$.

1. $\frac{d}{dx}[\ln x] = \frac{1}{x}, \quad x > 0$

2. $\frac{d}{dx}[\ln u] = \frac{1}{u} \frac{du}{dx} = \frac{u'}{u}, \quad u > 0$
Example  Differentiation of Logarithmic Functions

a. \[ \frac{d}{dx}[\ln(2x)] = \frac{u'}{u} \]
   \[ = \frac{2}{2x} \]
   \[ = \frac{1}{x} \]
   \[ u = 2x \]

b. \[ \frac{d}{dx}[\ln(x^2 + 1)] = \frac{u'}{u} \]
   \[ = \frac{2x}{x^2 + 1} \]
   \[ u = x^2 + 1 \]

Example – Differentiation of Logarithmic Functions  cont’d

c. \[ \frac{d}{dx}[x \ln x] = x \left( \frac{d}{dx} \ln x \right) + \ln x \left( \frac{d}{dx} x \right) \]
   \[ = x \left( \frac{1}{x} \right) + (\ln x)(1) \]
   \[ = 1 + \ln x \]

d. \[ \frac{d}{dx}[(\ln x)^3] = 3(\ln x)^2 \frac{d}{dx} \ln x \]
   \[ = 3(\ln x)^2 \frac{1}{x} \]
   \[ \text{Chain Rule} \]
The Derivative of the Natural Logarithmic Function

Because the natural logarithm is undefined for negative numbers, you will often encounter expressions of the form $\ln|u|$.

Theorem 3.14 states that you can differentiate functions of the form $y = \ln|u|$ as if the absolute value notation was not present.

**THEOREM 3.14 DERIVATIVE INVOLVING ABSOLUTE VALUE**

If $u$ is a differentiable function of $x$ such that $u \neq 0$, then

$$\frac{d}{dx} [\ln|u|] = \frac{u'}{u}.$$

Find the derivatives

a) $f(x) = \sin(x^2 + x)$

b) $h(x) = \cos 2x$

c) $g(t) = \tan(5 - \sin 2t)$
Find the derivative:

a) \[ y = \frac{\sin^2 x}{\cos x} \]

Use the Chain Rule to find the derivative of the following functions.

\[ y = (3x^2 + 7x)^{10} \]

\[ y = 5(7x^3 + 1)^{-3} \]
$y = \tan(3x + 1)$

$y = \sin(4x^3 + 3x + 1)$

$y = \theta^2 \sec 5\theta$

$y = (\sec x + \tan x)^5$
\[ y = \cos^3 x \]

\[ y = \cos x^3 \]

\[ y = \sqrt{1 + \cot^2 x} \]

\[ y = \sin^5(\cos 3x) \]
Find the second derivative of each of the following functions.

\[ y = \sin x^2 \]

\[ y = (x^2 + 1)^{-2} \]
Find the derivative of the function.

\[ f(x) = e^3 \ln x \]

\[ h(x) = \ln(2x^2 + 3) \]
$f(x) = \ln\left(\frac{2x}{x+3}\right)$

Find the equation of the line tangent to $y = \sec 2x$ at $x = \pi/6$. 
Find an equation of the line tangent to the graph of

\[ y = \frac{(x^2 - 1)^2}{x^3 - 6x - 1} \] at point (3,8).