Use the properties of exponents to simplify the expressions.

\[
\left( \frac{1}{e} \right)^{-2} \\
\left( \frac{e^5}{e^2} \right)^{-1} \\
\left( e^7 \right)^3 \\
e^3 \left( e^6 \right)
\]
Example – Sketching Graphs of Exponential Functions

• Sketch the graphs of the functions

\[ f(x) = 2^x, \quad g(x) = \left(\frac{1}{2}\right)^x = 2^{-x}, \quad \text{and} \quad h(x) = 3^x. \]

• To sketch the graphs of these functions by hand, you can complete a table of values, plot the corresponding points, and connect the points with smooth curves.

<table>
<thead>
<tr>
<th>( x )</th>
<th>(-3)</th>
<th>(-2)</th>
<th>(-1)</th>
<th>(0)</th>
<th>(1)</th>
<th>(2)</th>
<th>(3)</th>
<th>(4)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 2^x )</td>
<td>(\frac{1}{8})</td>
<td>(\frac{1}{4})</td>
<td>(\frac{1}{2})</td>
<td>(1)</td>
<td>(2)</td>
<td>(4)</td>
<td>(8)</td>
<td>(16)</td>
</tr>
<tr>
<td>( 2^{-x} )</td>
<td>(8)</td>
<td>(4)</td>
<td>(2)</td>
<td>(1)</td>
<td>(\frac{1}{2})</td>
<td>(\frac{1}{4})</td>
<td>(\frac{1}{8})</td>
<td>(\frac{1}{16})</td>
</tr>
<tr>
<td>( 3^x )</td>
<td>(\frac{1}{27})</td>
<td>(\frac{1}{9})</td>
<td>(\frac{1}{3})</td>
<td>(1)</td>
<td>(3)</td>
<td>(9)</td>
<td>(27)</td>
<td>(81)</td>
</tr>
</tbody>
</table>
The Number $e$

• In calculus, the natural (or convenient) choice for a base of an exponential number is the irrational number $e$, whose decimal approximation is

  \[ e \approx 2.71828182846. \]

• This choice may seem anything but natural. However, the convenience of this particular base will become apparent as you continue in this course.

Example – *Investigating the Number $e$*

• Use a graphing utility to graph the function

  \[ f(x) = (1 + x)^{1/x}. \]

• Describe the behavior of the function at values of $x$ that are close to 0.

• Solution:

  One way to examine the values of $f(x)$ near 0 is to construct a table.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$-0.01$</th>
<th>$-0.001$</th>
<th>$-0.0001$</th>
<th>$0.0001$</th>
<th>$0.001$</th>
<th>$0.01$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(1 + x)^{1/x}$</td>
<td>2.7320</td>
<td>2.7196</td>
<td>2.7184</td>
<td>2.7181</td>
<td>2.7169</td>
<td>2.7048</td>
</tr>
</tbody>
</table>
Example Solution

• From the table, it appears that the closer \( x \) gets to 0, the closer \((1 + x)^{1/x}\) gets to \( e \). You can confirm this by graphing the function \( f \), as shown in Figure 1.48.

![Graph of \( f(x) = (1 + x)^{1/x} \)](image)

Example Solution

• Try using a graphing calculator to obtain this graph. Then zoom in closer and closer to \( x = 0 \). Although \( f \) is not defined when \( x = 0 \), it is defined for \( x \)-values that are arbitrarily close to zero.

• By zooming in, you can see that the value of \( f(x) \) gets closer and closer to \( e \approx 2.71828182846 \) as \( x \) gets closer and closer to 0.
Example Solution

• Later, when you study limits, you will learn that this result can be written as

$$\lim_{x \to 0} (1 + x)^{1/x} = e$$

• which is read as “the limit of \((1 + x)^{1/x}\) as \(x\) approaches 0 is \(e\).”

The Natural Logarithmic Function

• Because the natural exponential function \(f(x) = e^x\) is one-to-one, it must have an inverse function. Its inverse is called the natural logarithmic

<table>
<thead>
<tr>
<th>DEFINITION OF THE NATURAL LOGARITHMIC FUNCTION</th>
</tr>
</thead>
</table>
| Let \(x\) be a positive real number. The natural logarithmic function, denoted by \(\ln x\), is defined as follows. (\(\ln x\) is read as “el en of \(x\)” or “the natural log of \(x\)”)
| \(\ln x = b \quad \text{if and only if} \quad e^b = x.\) |
The Natural Logarithmic Function

• This definition tells you that a logarithmic equation can be written in an equivalent exponential form, and vice versa.
• Here are some examples.

<table>
<thead>
<tr>
<th>Logarithmic Form</th>
<th>Exponential Form</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \ln 1 = 0 )</td>
<td>( e^0 = 1 )</td>
</tr>
<tr>
<td>( \ln e = 1 )</td>
<td>( e^1 = e )</td>
</tr>
<tr>
<td>( \ln e^{-1} = -1 )</td>
<td>( e^{-1} = \frac{1}{e} )</td>
</tr>
</tbody>
</table>

Properties of Logarithms

Let \( x, y, \) and \( z \) be real numbers such that \( x > 0 \) and \( y > 0 \)

1. \( \ln xy = \ln x + \ln y \)

2. \( \ln \frac{x}{y} = \ln x - \ln y \)

3. \( \ln x^z = z \ln x \)
Example 5 – *Expanding Logarithmic Expressions*

• **a.** \( \ln \frac{10}{9} = \ln 10 - \ln 9 \)  
  Property 2

• **b.** \( \ln \sqrt{3x + 2} = \ln (3x + 2)^{1/2} \)  
  Rewrite with rational exponent.  
  \[ = \frac{1}{2} \ln (3x + 2) \]  
  Property 3

• **c.** \( \ln \frac{6x}{5} = \ln (6x) - \ln 5 \)  
  Property 2  
  \[ = \ln 6 + \ln x - \ln 5 \]  
  Property 1

---

*Expanding Logarithmic Expressions*  
cont’d

• **d.** \( \ln \frac{(x^2 + 3)^2}{\sqrt[3]{x^2 + 1}} = \ln (x^2 + 3)^2 - \ln (\sqrt[3]{x^2 + 1}) \)
  \[ = 2 \ln (x^2 + 3) - \left[ \ln x + \ln (x^2 + 1)^{1/3} \right] \]
  \[ = 2 \ln (x^2 + 3) - \ln x - (x^2 + 1)^{1/3} \]
  \[ = 2 \ln (x^2 + 3) - \ln x - \frac{1}{3} \ln (x^2 + 1) \]
Evaluate the expression.

\[ 4^{1/2} \]

\[ 8^{2/3} \]

\[ 64^{-1/2} \]

\[ \left( \frac{1}{4} \right)^3 \]

Solve for x.

\[ 4^x = 64 \]

\[ 5^{x+1} = 125 \]

\[ \left( \frac{1}{5} \right)^{2x} = 625 \]

\[ (x + 3)^{4/3} = 16 \]
Find the domain of the function.

\[ f(x) = \frac{1}{2 - e^x} \]

Apply the inverse properties of \( \ln x \) and \( e^x \) to simplify the given expression.

\[ \ln e^{2x-1} - 8 + e^{\ln x^3} \]
Use the properties of logarithms to expand the logarithmic expression.

\[ \ln \sqrt{x^5} \]
\[ \ln(xyz) \]
\[ \ln \frac{1}{e} \]

Write the expression as the logarithm of a single quantity.

\[ \frac{1}{3} \left[ 2 \ln(x + 3) + \ln x - \ln(x^2 - 1) \right] \]