MTH 113 Section 5.5
Trigonometric Equations

Figure 5.7 The equation $\sin x = \frac{1}{2}$ has five solutions when $x$ is restricted to the interval $\left[ -\frac{3\pi}{2}, \frac{7\pi}{2} \right]$. 
Equations Involving A Single Trigonometric Function

Solve \( \cos x = \frac{\sqrt{2}}{2} \)

\( x = \frac{\pi}{4} \) is one solution

There are two solution between 0 and 2\( \pi \).

They are \( \frac{\pi}{4} \) and \( \frac{7\pi}{4} \).

But, any angle that is co-terminal with \( \pi/4 \) or \( 7\pi/4 \) has cosine equal to \( \frac{\sqrt{2}}{2} \)

All solutions to this equation are:

\( x = \frac{\pi}{4} + 2n\pi \) and \( x = \frac{7\pi}{4} + 2n\pi \), where \( n \) is any integer.

Find all solutions of each equation.

\( \cos x = \frac{\sqrt{3}}{2} \)

\( 2\sin x + \sqrt{3} = 0 \)
5 \sin \theta + 1 = 3 \sin \theta

7 \cos \theta + 9 = -2 \cos \theta

Solve for x where \(0 \leq x < 2\pi\):  \(3 \sin x = \sqrt{3} + \sin x\)

Solve for x where \(0 \leq x < 2\pi\):  \(\sin x + \cos x = 0\)
Solve for \( x \) where \( 0 \leq x < 2\pi \): \( \sin x \tan x = \sin x \)

**Equations Involving Multiple Angles**

Solve the equation: \( \tan 3x = 1, \ 0 \leq x < 2\pi \).

**Solution** The period of the tangent function is \( \pi \). In the interval \( [0, \pi) \), the only value for which the tangent function is 1 is \( \frac{\pi}{4} \). This means that \( 3x = \frac{\pi}{4} \). Because the period is \( \pi \), all the solutions to \( 3x = 1 \) are given by

\[
3x = \frac{\pi}{4} + n\pi \quad n \text{ is any integer.}
\]

Thus,

\[
x = \frac{\pi}{12} + \frac{n\pi}{3}
\]

Divide both sides by 3 and solve for \( x \).

In the interval \([0, 2\pi]\), we obtain the solutions of \( \tan 3x = 1 \) as follows:

\[
\begin{align*}
\text{Let } n = 0, & \quad x = \frac{\pi}{12} + \frac{0\pi}{3} \\
\text{Let } n = 1, & \quad x = \frac{\pi}{12} + \frac{1\pi}{3} \\
\text{Let } n = 2, & \quad x = \frac{\pi}{12} + \frac{2\pi}{3} \\
\end{align*}
\]

\[
\begin{align*}
\text{Let } n = 0, & \quad x = \frac{\pi}{12} + \frac{3\pi}{3} \\
\text{Let } n = 1, & \quad x = \frac{\pi}{12} + \frac{4\pi}{3} \\
\text{Let } n = 2, & \quad x = \frac{\pi}{12} + \frac{5\pi}{3} \\
\end{align*}
\]

\[
\begin{align*}
\text{Let } n = 0, & \quad x = \frac{\pi}{12} + \frac{12\pi}{3} \\
\text{Let } n = 1, & \quad x = \frac{\pi}{12} + \frac{16\pi}{3} \\
\text{Let } n = 2, & \quad x = \frac{\pi}{12} + \frac{20\pi}{3} \\
\end{align*}
\]

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Solve for $x$: \[ \tan 2x = \sqrt{3}, \quad 0 \leq x < 2\pi \]

Solve each equation on the interval $[0, 2\pi]$.

\[ \cos 2x = \frac{\sqrt{2}}{2} \]
\[
\tan \frac{x}{2} = \frac{\sqrt{3}}{3}
\]

\[
\cos \frac{2\theta}{3} = -1
\]
\[ \sin \left( 2x - \frac{\pi}{4} \right) = \frac{\sqrt{2}}{2} \]

**Trigonometric Equations Quadratic in Form**

Solve the equation: \( 2 \cos^2 x + \cos x - 1 = 0, \ 0 \leq x < 2\pi. \)

**Solution** The given equation is in quadratic form \( 2u^2 + u - 1 = 0 \) with \( u = \cos x \). Let us attempt to solve the equation by factoring.

\[
\begin{align*}
2 \cos^2 x + \cos x - 1 &= 0 \\
(2 \cos x - 1)(\cos x + 1) &= 0
\end{align*}
\]

This is the given equation. Factor. Notice that \((2u - 1)(u + 1)\).

Set each factor equal to 0.

\[
\begin{align*}
2 \cos x - 1 &= 0 & \text{or} & & \cos x + 1 &= 0 \\
2 \cos x &= 1 & & \cos x &= -1 \\
\cos x &= \frac{1}{2} & & \text{Solve for } \cos x.
\end{align*}
\]

\[
\begin{align*}
x &= \frac{\pi}{3} & x &= 2\pi - \frac{\pi}{3} = \frac{5\pi}{3} & x &= \pi
\end{align*}
\]

Solve each equation for \( x \), \( 0 \leq x < 2\pi \).

The solutions in the interval \([0, 2\pi]\) are \( \frac{\pi}{3} \), \( \pi \), and \( \frac{5\pi}{3} \).
\[ 2 \sin^2 x + \sin x - 1 = 0 \]

\[ \cos^2 + 2 \cos x - 3 = 0 \]
\[ \cos^2 - 1 = 0 \]

\[ 3 \tan^2 x - 9 = 0 \]
$4 \sec^2 x - 2 = 0$

$(\tan x + 1)(\sin x - 1) = 0$
\[ \cot x (\tan x + 1) = 0 \]

\[ \cos x - 2 \sin x \cos x = 0 \]
Use an identity to solve each equation on the interval \([0, 2\pi]\).

\[2 \cos^2 x - \sin x - 1 = 0\]

\[3 \cos^2 x = \sin^2 x\]
\[
\sin 2x = \sin x
\]

\[
\cos 2x + \cos x + 1 = 0
\]
\[
\sin x + \cos x = -1
\]

\[
\sin \left( x + \frac{\pi}{3} \right) + \sin \left( x - \frac{\pi}{3} \right) = 1
\]
Use a calculator to solve correct to four decimal places in $[0, 2\pi]$. Use radian measure.

\[
\sin x = 0.7392
\]

\[
3\cos^2 x - 8\cos x - 3 = 0
\]

\[
2\sin^2 x + 3\sin x - 4 = 0
\]