MORE ON TRIANGLES
Trigonometric Functions of Any Angle

We know how to find the trig function values of an acute angle from a right triangle. Let the point \( P = (x, y) \) be a point “\( r \)” units from the origin on the terminal side of angle \( \theta \). A right triangle can be formed by drawing a perpendicular line from \( P \) to the x-axis.

Below are examples of the right triangles formed from the angle \( \theta \). Each example has \( \theta \) in a different one of the four quadrants in the \( xy \)-plane.

We know how to use right triangles to evaluate the trigonometric functions of an acute angle. However, in the figures above the angle \( \theta \) is not always an acute angle. We will often evaluate trigonometric functions of positive angles greater than 90° and all negative angles by making use of a positive acute angle. This positive acute angle is called a \textit{REFERENCE ANGLE}.

Definition of Reference Angle – Let \( \theta \) be a non-acute angle in standard position that lies in a quadrant. Its reference angle is the positive acute angle \( \alpha \) formed by the terminal side of the angle \( \theta \) and the \( x \)-axis.

Below are examples of the right triangles formed by the angle \( \theta \) with the reference angle, \( \alpha \), labeled.
Examples: Find the reference angle $\alpha$ for each of the following angles $\theta$.

(i) $\theta = 60^\circ$ then $\alpha =$ __________

(ii) $\theta = 135^\circ$ then $\alpha =$ __________

(iii) $\theta = 210^\circ$ then $\alpha =$ __________

(iv) $\theta = -45^\circ$ then $\alpha =$ __________

Once again we have our examples of the right triangles formed by the angle $\theta$ with the reference angle, $\alpha$, labeled.

Notice that for each right triangle, the length of the side opposite $\alpha$ is $y$ (the $y$-coordinate of the point $P$). The length of the side adjacent to $\alpha$ is $x$ (the $x$-coordinate of the point $P$). Using Pythagorean’s Theorem, we can find the length of the hypotenuse, $r$. So the trigonometric functions evaluated at $\alpha$ are defined as:

\[
\sin \alpha = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{y}{r} \quad \cos \alpha = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{x}{r} \quad \tan \alpha = \frac{\text{opposite}}{\text{adjacent}} = \frac{y}{x}
\]

\[
\csc \alpha = \frac{\text{hypotenuse}}{\text{opposite}} = \frac{r}{y} \quad \sec \alpha = \frac{\text{hypotenuse}}{\text{adjacent}} = \frac{r}{x} \quad \cot \alpha = \frac{\text{adjacent}}{\text{opposite}} = \frac{x}{y}
\]
Using Reference angles to evaluate trigonometric Functions

The values of the trigonometric functions of a given angle, $\theta$, are the same as the values of the trigonometric functions of the reference angle, $\alpha$, except for possibly the sign. A function value of the acute reference angle $\alpha$ is always positive. However, the same function value for $\theta$ may be positive or negative.

**Example:*** Let the point $P = (x, y) = (3, -4)$ be on the terminal side of the angle $\theta$.
Find $\sin \theta$ and $\cos \theta$.

Using Pythagorean’s Theorem and the facts that we know from right triangles:

\[
x^2 + y^2 = r^2\]

\[
3^2 + 4^2 = r^2
\]
\[
9 + 16 = r^2
\]
\[
25 = r^2
\]
\[
r = 5
\]

Recall that when using the reference angle $\alpha$ the $x$, $y$, and $r$ values are all positive. So,

\[
\sin \alpha = \frac{y}{r} = \frac{4}{5} \quad \text{ and } \quad \cos \alpha = \frac{x}{r} = \frac{3}{5}.
\]

Since $\theta$ lies in quadrant IV, the $y$-coordinate is negative. So, $\sin \theta = -\sin \alpha = \frac{-4}{5}$.

However, also since $\theta$ lies in quadrant IV, the $x$-coordinate is positive. So, $\cos \theta = \cos \alpha = \frac{3}{5}$.

**Example 2:** Find the $\sin \theta$ and $\cos \theta$ for $\theta = 150^\circ$.

When the reference angle, $\alpha$, is a common angle ($30^\circ$, $45^\circ$, $60^\circ$) we do not need to use Pythagorean’s Theorem to help us establish a right triangle. Instead the right triangle defined by the reference angle will be one of our basic triangles.

If $\theta = 150^\circ$ then the reference will be $\alpha = 30^\circ$.
So the right triangle is a $30^\circ - 60^\circ - 90^\circ$ triangle:

\[
y = \frac{1}{2}
\]
\[
x = \frac{\sqrt{3}}{2}
\]

Recall that when using the reference angle $\alpha$ the $x$, $y$, and $r$ values are all positive. So,

\[
\sin \alpha = \frac{y}{r} = \frac{1/2}{1} = \frac{1}{2} \quad \text{ and } \quad \cos \alpha = \frac{x}{r} = \frac{\sqrt{3}/2}{1} = \frac{\sqrt{3}}{2}.
\]

Since $\theta$ lies in quadrant II, the $y$-coordinate is positive. So,

\[
\sin \theta = \sin \alpha = \frac{+y}{r} = \frac{1/2}{1} = \frac{1}{2}.
\]

However, also since $\theta$ lies in quadrant II, the $x$-coordinate is negative. So,

\[
\cos \theta = -\cos \alpha = \frac{-x}{r} = -\frac{\sqrt{3}/2}{1} = -\frac{\sqrt{3}}{2}.
\]

Reference for information: 
Algebra & Trigonometry 2nd Ed. 
by Robert Blitzer