The Law of Cosines

If \( A, B, \) and \( C \) are the measures of the angles of a triangle, and \( a, b, \) and \( c \) are the lengths of the sides opposite these angles, then

\[
\begin{align*}
a^2 &= b^2 + c^2 - 2bc \cos A \\
b^2 &= a^2 + c^2 - 2ac \cos B \\
c^2 &= a^2 + b^2 - 2ab \cos C.
\end{align*}
\]

The square of a side of a triangle equals the sum of the squares of the other two sides minus twice their product times the cosine of their included angle.

Deriving the Law of Cosines

Using the right triangle that contains angle \( \angle A \), we apply the definitions of the cosine and the sine.

\[
\begin{align*}
\cos A &= \frac{x}{b} \\
\sin A &= \frac{y}{b} \\
x &= b \cos A \\
y &= b \sin A
\end{align*}
\]

Multiply both sides of each equation by \( b \) and solve for \( x \) and \( y \), respectively.

We now apply the distance formula to the side of the triangle with length \( a \). Notice that \( a \) is the distance from \((x, y)\) to \((c, 0)\).

\[
\begin{align*}
a &= \sqrt{(x - c)^2 + (y - 0)^2} \\
a^2 &= (x - c)^2 + y^2 \\
a^2 &= (b \cos A - c)^2 + (b \sin A)^2 \\
a^2 &= b^2 \cos^2 A - 2bc \cos A + c^2 + b^2 \sin^2 A \\
a^2 &= b^2 \sin^2 A + b^2 \cos^2 A + c^2 - 2bc \cos A \\
a^2 &= b^2 \sin^2 A + b^2 \cos^2 A + c^2 - 2bc \cos A \\
a^2 &= b^2 + c^2 - 2bc \cos A \\
\end{align*}
\]

Use the distance formula.
Square both sides of the equation.
Rearrange terms.
Factor \( b^2 \) from the first two terms.

\[
\sin^2 A + \cos^2 A = 1
\]
Solving an SAS Triangle

1. Use the Law of Cosines to find the side opposite the given angle.
2. Use the Law of Sines to find the angle opposite the shorter of the two given sides. This angle is always acute.
3. Find the third angle by subtracting the measure of the given angle and the angle found in step 2 from 180°.

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Solve. Round lengths to nearest tenth and angle measures to nearest degree.

1) Find b using the Law of Cosines.

2) Find the angle opposite the shorter side using the Law of Sines.

3) Find the 3rd angle.
**Solving an SSS Triangle**

1. Use the Law of Cosines to find the angle opposite the longest side.
2. Use the Law of Sines to find either of the two remaining acute angles.
3. Find the third angle by subtracting the measures of the angles found in steps 1 and 2 from 180°.

1) Use the Law of Cosines to find the angle opposite the longest side.

2) Use the Law of Sines to find either of the other two angles.

3) Find the third side.
Application of the Law of Cosines

If you are on island B, what bearing should you navigate to go to island C?

If you are on island A, what bearing should you navigate to go to island C?

Solve the triangle.

A = 3
B = 2
C = c
A plane leaves airport A and travels 580 miles to airport B on a bearing of N34°E. The plane later leaves airport B and travels to airport C 400 miles away on a bearing of S74°E. Find the distance from airport A to airport C to the nearest tenth of a mile.

**Heron’s Formula for the Area of a Triangle**

The area of a triangle with sides $a$, $b$, and $c$ is

$$\text{Area} = \sqrt{s(s - a)(s - b)(s - c)},$$

where $s$ is one-half its perimeter: $s = \frac{1}{2}(a + b + c)$.

Find the area of the triangle described.

$a = 5$ feet, $b = 5$ feet, $c = 4$ feet
Bob Fernando's triangular property has the measurements of 40 yards, 50 yards and 30 yards. Find the area of the property.