**The Law of Sines**

If $A$, $B$, and $C$ are the measures of the angles of a triangle, and $a$, $b$, and $c$ are the lengths of the sides opposite these angles, then

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

The ratio of the length of the side of any triangle to the sine of the angle opposite that side is the same for all three sides of the triangle.

**Study Tip**

The Law of Sines can be expressed with the sines in the numerator:

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}.$$
Deriving the Law of Sines

\[
\sin B = \frac{h}{a} \quad \sin A = \frac{h}{b} \quad \sin \theta = \frac{\text{opposite}}{\text{hypotenuse}} \\
\frac{h - a \sin B}{a} = \frac{h - b \sin A}{b} \\
\text{Solve each equation for } h
\]

Because we have found two expressions for \( h \), we can set these expressions equal to each other.

\[
\frac{a \sin B}{\sin A} = \frac{b \sin A}{\sin B} \quad \text{Equate the expressions for } h, \quad \text{Divide both sides by } \sin A \sin B.
\]

\[
\frac{a}{\sin A} = \frac{b}{\sin B} \quad \text{Simplify.}
\]

This proves part of the Law of Sines. If we use the same process and draw an altitude of length \( h \) from vertex \( A \), we obtain the following result:

\[
\frac{b}{\sin B} = \frac{c}{\sin C}
\]

Study Tip

Up until now, our work with triangles has involved right triangles. Do not apply relationships that are valid for right triangles to oblique triangles. Avoid the error of using the Pythagorean Theorem, \( a^2 + b^2 = c^2 \), to find a missing side of an oblique triangle. This relationship among the three sides applies only to right triangles.

An oblique triangle is a triangle that does not contain a right angle.
Solving Oblique Triangles

Some oblique triangles can be solved by using the Law of Sines.

Solve the triangle shown in the picture. $A=20^\circ$, $C = 43^\circ$, and $c$ is 32 inches. Round your answer to the nearest hundredth.

\[
\frac{a}{\sin 20^\circ} = \frac{32}{\sin 43^\circ}
\]
\[
a = \frac{32 \sin 20^\circ}{\sin 43^\circ}
\]
\[
a = 16.05 \text{ inches}
\]

Solve triangle ABC shown in the picture if $A=34^\circ$, $C = 72^\circ$, $b = 70$.

Sum of the Angles in a Triangle

\[
A + B + C = 180
\]
\[
34 + B + 72 = 180
\]
\[
B=74
\]

Law of Sines

\[
\frac{c}{\sin 72^\circ} = \frac{70}{\sin 74^\circ}
\]
\[
c = \frac{70 \sin 72^\circ}{\sin 74^\circ}
\]
\[
c = 69.26
\]
Solve the given triangle.

A = 6°, B = 12°, c = 100
The Ambiguous Case (SSA)

Consider a triangle in which \( a, b, \) and \( A \) are given. This information may result in:

- **One Triangle**
  - \( a > \hat{b} \) and \( a > \hat{b} \)
  - \( a = \hat{b} \) and is just the right length to form a right triangle.
- **No Triangle**
  - \( a < \hat{b} \) and is not long enough to form a triangle.
- **Two Triangles**
  - \( a < \hat{b} \) and \( a < \hat{b} \)

Example:

\[ a = 30, \ \hat{b} = 20, \ \hat{A} = 50^\circ \]
a = 10, b = 30, A = 150°

a = 95, c = 125, A = 49°
Area of An Oblique Triangle

The area of a triangle equals one-half the product of the lengths of two sides times the sine of their included angle. In Figure 6.10, this wording can be expressed by the formulas

\[ \text{Area} = \frac{1}{2}be \sin A = \frac{1}{2}ab \sin C = \frac{1}{2}ac \sin B. \]
Find the area of the given triangle.

Find the area of the given triangle.

\[ B = 125^\circ, \ a = 8 \text{ yards}, \ c = 5 \text{ yards} \]
A pine tree growing on a hillside makes a $75^\circ$ angle with the hill. From a point 80 feet up the hill, the angle of elevation to the top of the tree is $62^\circ$ and the angle of depression to the bottom is $23^\circ$. Find, to the nearest tenth of a foot, the height of the tree.