1.6 Transformation of Functions

Objective 1: Recognize graphs of common functions.

There are several functions whose graphs and characteristics need to be known, without hesitation, in order to be able to analyze their transformations into more complicated graphs.

- **Constant Function**
  - Domain: \((-\infty, \infty)\)
  - Range: the single number \(c\)
  - Constant on \((-\infty, \infty)\)
  - Even function

- **Identity Function**
  - Domain: \((-\infty, \infty)\)
  - Range: \((-\infty, \infty)\)
  - Increasing on \((-\infty, \infty)\)
  - Odd function
**Absolute Value Function**
- Domain: \((-\infty, \infty)\)
- Range: \([0, \infty)\)
- Decreasing on \((-\infty, 0)\) and increasing on \((0, \infty)\)
- Even function

**Standard Quadratic Function**
- Domain: \((-\infty, \infty)\)
- Range: \([0, \infty)\)
- Decreasing on \((-\infty, 0)\) and increasing on \((0, \infty)\)
- Even function

**Square Root Function**
- Domain: \([0, \infty)\)
- Range: \([0, \infty)\)
- Increasing on \((0, \infty)\)
- Neither even nor odd

**Standard Cubic Function**
- Domain: \((-\infty, \infty)\)
- Range: \((-\infty, \infty)\)
- Increasing on \((-\infty, \infty)\)
- Odd function
The function $f(x) = \sqrt[3]{x}$ has the following properties:

- **Domain:** $(-\infty, \infty)$
- **Range:** $(-\infty, \infty)$
- **Increasing on:** $(-\infty, \infty)$
- **Odd function**

The function $f(x) = \frac{1}{x^2}$ has the following properties:

- **Domain:** $(-\infty, 0) \cup (0, \infty)$
- **Range:** $(0, \infty)$
- **Increasing:** $(-\infty, 0)$
- **Decreasing:** $(0, \infty)$
- **Even function**
Objective 2: Use vertical shifts to graph functions.

Vertical Shifts
Let $f$ be a function and $c$ be a positive real number.
- The graph of $y = f(x) + c$ is the graph of $y = f(x)$ shifted $c$ units vertically upward.
- The graph of $y = f(x) - c$ is the graph of $y = f(x)$ shifted $c$ units vertically downward.
Use the graph of \( f(x) = |x| \) to obtain the graph of \( g(x) = |x| - 4 \).

**Solution** The graph of \( g(x) = |x| - 4 \) has the same shape as the graph of \( f(x) = |x| \). However, it is shifted down vertically 4 units.

It is easier to correct transformations if you first identify some of the points on the given function’s graph.
Objective 3: Use horizontal shifts to graph functions.

Horizontal Shifts
Let $f$ be a function and $c$ a positive real number.

- The graph of $y = f(x + c)$ is the graph of $y = f(x)$ shifted to the left $c$ units.
- The graph of $y = f(x - c)$ is the graph of $y = f(x)$ shifted to the right $c$ units.
Use the graph of $f(x) = x^2$ to obtain $g(x) = (x+1)^2$.

Combining Horizontal and Vertical Shifts
Use the graph of \( f(x) = x^2 \) to obtain \( g(x) = (x+1)^2 + 2 \)

Objective 4: Use reflections to graph functions.

Reflection about the \( x \)-Axis

The graph of \( y = -f(x) \) is the graph of \( y = f(x) \) reflected about the \( x \)-axis.
Reflections about the x-axis

Reflection about the y-Axis
The graph of \( y = f(-x) \) is the graph of \( y = f(x) \) reflected about \( y \)-axis.

\[
g(x) = -\sqrt{x} = -f(x).
\]
Use the graph of $f(x)=x^3$ to obtain the graph of $g(x)=(-x)^3$.

Use the graph of $f(x)=\sqrt{x}$ to graph $g(x)=\sqrt{-x}$.
Use the graph of \( f(x) = \sqrt{x} \) to graph \( g(x) = -\sqrt{x} \)

**Objective 5:** Use vertical stretching and shrinking to graph functions.

**Vertically Stretching and Shrinking Graphs**

Let \( f \) be a function and \( c \) a positive real number.
- If \( c > 1 \), the graph of \( y = cf(x) \) is the graph of \( y = f(x) \) vertically stretched by multiplying each of its \( y \)-coordinates by \( c \).
- If \( 0 < c < 1 \), the graph of \( y = cf(x) \) is the graph of \( y = f(x) \) vertically shrunk by multiplying each of its \( y \)-coordinates by \( c \).
Vertically Shrinking

Solution  The graph of \( h(x) = \frac{1}{3}x^3 \) is obtained by vertically shrinking the graph of \( f(x) = x^3 \).

Vertically Stretching

This is vertical stretching – each y coordinate is multiplied by 3 to stretch the graph.
Use the graph of \( f(x) = |x| \) to graph \( g(x) = 2|x| \)

Objective 6: Use horizontal stretching and shrinking to graph functions.

**Horizontally Stretching and Shrinking Graphs**

Let \( f \) be a function and \( c \) a positive real number.

- If \( c > 1 \), the graph of \( y = f(cx) \) is the graph of \( y = f(x) \) horizontally shrunk by dividing each of its \( x \)-coordinates by \( c \).
- If \( 0 < c < 1 \), the graph of \( y = f(cx) \) is the graph of \( y = f(x) \) horizontally stretched by dividing each of its \( x \)-coordinates by \( c \).
Horizontal Shrinking

The graph of $y = f(x)$ with five points identified

Graph $y = f(x)$.

Horizontal shrinking the graph of $y = f(x)$.

Divide each x-coordinate by 2.

The graph of $g(x) = f(2x)$

Horizontal Stretching

The graph of $y = f(x)$ with five points identified

Graph $y = f(x)$.

The graph of $x = f(v)$ with two points identified

Graph $h(x) = f(\frac{1}{2}x)$.

Horizontally stretch the graph of $y = f(x)$.

Divide each x-coordinate by 2, which is the same as multiplying by 2.

The graph of $h(x) = f(\frac{1}{2}x)$
Use the graph of \( f(x) = |x| \) to obtain the graph of \( g(x) = \frac{1}{3}x \).

Objective 7: Graph functions involving a sequence of transformations.

A function involving more than one transformation can be graphed by performing transformations in the following order:

1. Horizontal shifting
2. Stretching or shrinking
3. Reflecting
4. Vertical shifting
A Sequence of Transformations

Starting graph.

Move the graph to the left 3 units.

Stretch the graph vertically by 2.

Shift down 1 unit.

Given the graph of f(x) below, graph \( \frac{1}{2} f(x - 1) \).
Given the graph of $f(x)$ below, graph $-f(x + 2) - 1$.

Given the graph of $f(x)$ below, graph $2f(-x) - 1$. 
Write the equation of the given graph \( g(x) \). The original function was \( f(x) = x^2 \)

(a) \( g(x) = (x + 4)^2 - 3 \)
(b) \( g(x) = (x - 4)^2 - 3 \)
(c) \( g(x) = (x + 4)^2 + 3 \)
(d) \( g(x) = (x - 4)^2 + 3 \)

Write the equation of the given graph \( g(x) \). The original function was \( f(x) = |x| \)

(a) \( g(x) = -|x - 4| \)
(b) \( g(x) = |x - 4| \)
(c) \( g(x) = -|x| + 4 \)
(d) \( g(x) = -|x| - 4 \)