3.5 Implicit and Explicit Functions

Up to this point, most functions have been expressed in **explicit form**. For example, in the equation

\[ y = 3x^2 - 5 \]

the variable \( y \) is explicitly written as a function of \( x \). Some functions, however, are only **implied** by an equation.

For instance, the function \( y = 1/x \) is defined **implicitly** by the equation \( xy = 1 \).

Suppose you were asked to find \( dy/dx \) for this equation. You could begin by writing \( y \) explicitly as a function of \( x \) and then differentiating.

### Implicit and Explicit Functions

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<td>( xy = 1 )</td>
<td>( y = \frac{1}{x} = x^{-1} )</td>
<td>( \frac{dy}{dx} = -x^{-2} = -\frac{1}{x^2} )</td>
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This strategy works whenever you can solve for the function explicitly.

You cannot, however, use this procedure when you are unable to solve for \( y \) as a function of \( x \).

For instance, how would you find \( dy/dx \) for the equation \( x^2 - 2y^3 + 4y = 2 \), where it is very difficult to express \( y \) as a function of \( x \) explicitly? To do this, you can use **implicit differentiation**.
Implicit and Explicit Functions

To understand how to find \( \frac{dy}{dx} \) implicitly, you must realize that the differentiation is taking place with respect to \( x \).

This means that when you differentiate terms involving \( x \) alone, you can differentiate as usual.

However, when you differentiate terms involving \( y \), you must apply the Chain Rule, because you are assuming that \( y \) is defined implicitly as a differentiable function of \( x \).

Example 1 – Differentiating with Respect to \( x \)

a. \[ \frac{d}{dx}[x^3] = 3x^2 \]
   
   Variables agree: use Simple Power Rule.

b. \[ \frac{d}{dx}[y^3] = 3y^2 \frac{dy}{dx} \]
   
   Variables disagree: use Chain Rule.

c. \[ \frac{d}{dx}[x + 3y] = 1 + 3 \frac{dy}{dx} \]
   
   Chain Rule: \[ \frac{d}{dx}[3y] = 3y' \]
Example 1 – Differentiating with Respect to x

d. \( \frac{d}{dx}[xy^2] = x \frac{d}{dx}[y^2] + y^2 \frac{d}{dx}[x] \)  
   \[= x \left(2y \frac{dy}{dx}\right) + y^2(1)\]  
   \[= 2xy \frac{dy}{dx} + y^2\]  

Product Rule  
Chain Rule  
Simplify.

Implicit Differentiation

**GUIDELINES FOR IMPLICIT DIFFERENTIATION**

1. Differentiate both sides of the equation with respect to \( x \).
2. Collect all terms involving \( \frac{dy}{dx} \) on the left side of the equation and move all other terms to the right side of the equation.
3. Factor \( \frac{dy}{dx} \) out of the left side of the equation.
4. Solve for \( \frac{dy}{dx} \) by dividing both sides of the equation by the left-hand factor that does not contain \( \frac{dy}{dx} \).

In Example 2, note that implicit differentiation can produce an expression for \( \frac{dy}{dx} \) that contains both \( x \) and \( y \).
Example 2 – Implicit Differentiation

Find \( \frac{dy}{dx} \) given that \( y^3 + y^2 - 5y - x^2 = -4 \).

Solution:

1. Differentiate both sides of the equation with respect to \( x \).

\[
\frac{d}{dx}[y^3 + y^2 - 5y - x^2] = \frac{d}{dx}[-4]
\]

\[
\frac{d}{dx}[y^3] + \frac{d}{dx}[y^2] - \frac{d}{dx}[5y] - \frac{d}{dx}[x^2] = \frac{d}{dx}[-4]
\]

\[
3y^2 \frac{dy}{dx} + 2y \frac{dy}{dx} - 5 \frac{dy}{dx} - 2x = 0
\]

Example 2 – Solution

cont’d

2. Collect the \( \frac{dy}{dx} \) terms on the left side of the equation.

\[
3y^2 \frac{dy}{dx} + 2y \frac{dy}{dx} - 5 \frac{dy}{dx} = 2x
\]

3. Factor \( \frac{dy}{dx} \) out of the left side of the equation.

\[
\frac{dy}{dx} (3y^2 + 2y - 5) = 2x
\]

4. Solve for \( \frac{dy}{dx} \) by dividing by \( (3y^2 + 2y - 5) \).

\[
\frac{dy}{dx} = \frac{2x}{3y^2 + 2y - 5}
\]
Find $dy/dx$ by implicit differentiation.

$$x^2 y + y^2 x = -2$$

$$(\sin \pi x + \cos \pi y)^2 = 2$$
Find $dy/dx$ by implicit differentiation and evaluate the derivative at the given point.

\[ y^3 - x^2 = 4 \quad (2, 2) \]

Use implicit differentiation to find an equation of the tangent line to the graph at the given point.

\[ y^2 + \ln xy = 2, \quad (e, 1) \]
Find \( d^2 y / dx^2 \) implicitly in terms of \( x \) and \( y \).

\[
x^2 y - 4x = 5
\]