2.3 Polynomial Functions and their Graph

Objective 1: Identify polynomial functions.

Definition of a Polynomial Function
Let \( n \) be a nonnegative integer and let \( a_n, a_{n-1}, \ldots, a_2, a_1, a_0 \) be real numbers, with \( a_n \neq 0 \). The function defined by

\[
f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0
\]

is called a polynomial function of degree \( n \). The number \( a_n \), the coefficient of the variable to the highest power, is called the leading coefficient.

Polynomial functions of degree 2 or higher have graphs that are smooth and continuous. By smooth, we mean that the graphs contain only rounded curves with no sharp corners. By continuous, we mean that the graphs have no breaks and can be drawn without lifting your pencil from the rectangular coordinate system.
Objective 2: Determine the end behavior of the graphs of polynomial functions.

The Leading Coefficient Test
As \( x \) increases or decreases without bound, the graph of the polynomial function

\[
f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \quad (a_n \neq 0)
\]
eventually rises or falls. In particular,

1. For \( n \) odd:
   - If the leading coefficient is positive, the graph falls to the left and rises to the right.
   - If the leading coefficient is negative, the graph rises to the left and falls to the right.

2. For \( n \) even:
   - If the leading coefficient is positive, the graph rises to the left and to the right.
   - If the leading coefficient is negative, the graph falls to the left and to the right.
Odd-degree polynomial functions have graphs with opposite behavior at each end. Even-degree polynomial functions have graphs with the same behavior at each end.

Use the Leading Coefficient Test to determine the end behavior of the graph of \( f(x) = -3x^3 - 4x + 7 \)

Example

Use the Leading Coefficient Test to determine the end behavior of the graph of \( f(x) = -0.08x^4 - 9x^3 + 7x^2 + 4x + 7 \)

This is the graph that you get with the standard viewing window. How do you know that you need to change the window to see the end behavior of the function? What viewing window will allow you to see the end behavior?
Objective 3: Use factoring to find zeros of polynomial functions.

If \( f \) is a polynomial function, then the values of \( x \) for which \( f(x) \) is equal to 0 are called the zeros of \( f \). These values of \( x \) are the roots, or solutions, of the polynomial equation \( f(x)=0 \). Each real root of the polynomial equation appears as an \( x \)-intercept of the graph of the polynomial function.

Find all zeros of \( f(x)= x^3+4x^2\ - \ 3x\ - \ 12 \)

Example

Find all zeros of \( x^3+2x^2\ - 4x\ - 8=0 \)
Objective 4: Identify zeros and their multiplicities.

**Multiplicity and x-Intercepts**

If \( r \) is a zero of even multiplicity, then the graph **touches** the x-axis and **turns around** at \( r \). If \( r \) is a zero of odd multiplicity, then the graph **crosses** the x-axis at \( r \). Regardless of whether the multiplicity of a zero is even or odd, graphs tend to flatten out at zeros with multiplicity greater than one.

Find the zeros of \( x^3 + 2x^2 - 4x - 8 = 0 \)

\[
(x^3 + 2x^2) + (-4x - 8) = 0 \\
x^2(x + 2) - 4(x + 2) = 0 \\
(x + 2)(x^2 - 4) = 0 \\
(x + 2)(x + 2)(x - 2) = 0 
\]

−2 has a multiplicity of 2, and 2 has a multiplicity of 1. Notice how the graph touches at -2 (even multiplicity), but crosses at 2 (odd multiplicity).
Graphing Calculator - Finding the Zeros

\[ x^3 + 2x^2 - 4x - 8 = 0 \]

The x-intercepts are the zeros of the function. To find the zeros, press 2nd Trace then #2. The zero -2 has multiplicity of 2.

Example

Find the zeros of \( f(x) = (x - 3)^2(x - 1)^3 \) and give the multiplicity of each zero. State whether the graph crosses the x-axis or touches the x-axis and turns around at each zero. Sketch the graph.
Objective 5: Use the Intermediate Value Theorem.

The Intermediate Value Theorem for Polynomials
Let $f$ be a polynomial function with real coefficients. If $f(a)$ and $f(b)$ have opposite signs, then there is at least one value of $c$ between $a$ and $b$ for which $f(c) = 0$. Equivalently, the equation $f(x) = 0$ has at least one real root between $a$ and $b$.

Show that the function $y = x^3 - x + 5$ has a zero between -2 and -1.

$$f(-2) = (-2)^3 - (-2) + 5 = -1$$
$$f(-1) = (-1)^3 - (-1) + 5 = 5$$

Since the signs of $f(-1)$ and $f(-2)$ are opposites then by the Intermediate Value Theorem there is at least one zero between $f(-2)$ and $f(-1)$. You can also see these values on the table below. Press 2nd Graph to get the table below.
Example

Show that the polynomial function \( f(x) = x^3 - 2x + 9 \) has a real zero between -3 and -2.

Objective 6: Understand the relationship between degree and turning points.

The graph of \( f(x) = x^5 - 6x^3 + 8x + 1 \) is shown below. The graph has four smooth turning points. The polynomial is of degree 5. Notice that the graph has four turning points. In general, if the function is a polynomial function of degree \( n \), then the graph has at most \( n-1 \) turning points.
Objective 7: Graph polynomial functions.

**Graphing a Polynomial Function**

\[ f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0, \quad a_n \neq 0 \]

1. Use the Leading Coefficient Test to determine the graph’s end behavior.
2. Find x-intercepts by setting \( f(x) = 0 \) and solving the resulting polynomial equation. If there is an x-intercept at \( r \) as a result of \( (x - r)^k \) in the complete factorization of \( f(x) \), then
   a. If \( k \) is even, the graph touches the x-axis at \( r \) and turns around.
   b. If \( k \) is odd, the graph crosses the x-axis at \( r \).
   c. If \( k > 1 \), the graph flattens out at \( (r, 0) \).
3. Find the y-intercept by computing \( f(0) \).
4. Use symmetry, if applicable, to help draw the graph:
   a. y-axis symmetry: \( f(-x) = f(x) \)
   b. Origin symmetry: \( f(-x) = -f(x) \).
5. Use the fact that the maximum number of turning points of the graph is \( n - 1 \) to check whether it is drawn correctly.

Graph \( f(x) = x^4 - 4x^2 \) using what you have learned in this section.

**x-intercepts:**

**End behavior:**

**Behavior at the x-intercepts:**

**y-intercept:**

**Symmetry:**

**Maximum number of turning points:**
Graph \( f(x) = x^3 - 9x^2 \) using what you have learned in this section.

End behavior:

<table>
<thead>
<tr>
<th>x-intercepts:</th>
</tr>
</thead>
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Behavior at the x-intercepts:

| y-intercept: |

Symmetry:

| Maximum number of turning points: |

Graph \( f(x) = (x + 3)(x + 1)^3(x + 4) \) using what you have learned in this section.

End behavior:

| x-intercepts: |

Behavior at the x-intercepts:

| y-intercept: |

Symmetry:

| Maximum number of turning points: |