2.2 Quadratic Functions

Objective 1: Recognize characteristics of parabolas.

Quadratic functions are any functions of the form 
\( f(x) = ax^2 + bx + c \) where \( a \neq 0 \), and \( a, b \) and \( c \) are 
real numbers. The graph of any quadratic 
function is called a parabola. Parabolas are 
shaped like cups. Parabolas are symmetric with 
respect to a line called the axis of symmetry. 
If a parabola is folded along its axis of symmetry, 
the two halves match exactly.

Graphs of Quadratic Functions

Parabolas

\[ f(x) = ax^2 + bx + c \]
Objective 2: Graph parabolas.

The Standard Form of a Quadratic Function

The quadratic function

\[ f(x) = a(x - h)^2 + k, \quad a \neq 0 \]

is in standard form. The graph of \( f \) is a parabola whose vertex is the point \((h, k)\). The parabola is symmetric with respect to the line \( x = h \). If \( a > 0 \), the parabola opens upward; if \( a < 0 \), the parabola opens downward.

Graphing Quadratic Functions with Equations in Standard Form

To graph \( f(x) = a(x - h)^2 + k \),

1. Determine whether the parabola opens upward or downward. If \( a > 0 \), it opens upward. If \( a < 0 \), it opens downward.
2. Determine the vertex of the parabola. The vertex is \((h, k)\).
3. Find any \( x \)-intercepts by solving \( f(x) = 0 \). The function’s real zeros are the \( x \)-intercepts.
4. Find the \( y \)-intercept by computing \( f(0) \).
5. Plot the intercepts, the vertex, and additional points as necessary. Connect these points with a smooth curve that is shaped like a cup.

Seeing the Transformations

**Figure 3.2(a)**
\( a > 0 \) Parabola opens upward.

**Figure 3.2(b)**
\( a < 0 \) Parabola opens downward.
**Using Standard Form**

\[ f(x) = -2(x - 3)^2 + 8 \]

**Vertex** \((h, k)\) \(V(3, 8)\)

**Axis of Symmetry** \(x = 3\)

Finding the x intercept, let \(y = 0\)

\[ 0 = -2(x - 3)^2 + 8 \]

\[ -8 \]

\[ -2 \]

\[ 4 = (x - 3)^2 \]

\[ \pm \sqrt{4} = \sqrt{(x - 3)^2} \]

\[ \pm 2 = x - 3 \]

\[ 3 \pm 2 = x, \quad (5, 0), (1, 0) \]

Finding the y intercept let \(x = 0\)

\[ y = -2(0 - 3)^2 + 8 \quad y = -10 \quad (0, -10) \]

\(a < 0\) so the parabola has a maximum, opens down.

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**Graph the quadratic function** \(f(x) = -(x + 2)^2 + 4\). **Use the graph to determine the function's domain and range.**

The vertex is \((-2, 4)\)

- The axis of symmetry is \(x = -2\).
- Find the x-intercepts.
  - \(0 = -(x + 2)^2 + 4\)
  - \(-4 = -(x + 2)^2\)
  - \(4 = (x + 2)^2\)
  - \(\pm \sqrt{4} = \sqrt{(x + 2)^2}\)
  - \(\pm 2 = x + 2\)
  - \(x + 2 = 2\) or \(x + 2 = -2\)
  - \(x = 0\) or \(x = -4\)
- \(x\) intercepts: \((0, 0)\) and \((-4, 0)\)

- Find the y-intercept.
  - \(y = -(0 + 2)^2 + 4\)
  - \(= -4 + 4 = 0\)
- The y-intercept is \((0, 0)\).
Graph the quadratic function \( f(x) = (x-3)^2 - 4 \). Use the graph to determine the function’s domain and range. The vertex is (3, -4)

The axis of symmetry is \( x = 3 \)

\[ (x-3)^2 = 4 \]
\[ \sqrt{(x-3)^2} = \pm\sqrt{4} \]
\[ x - 3 = -2 \text{ or } x-3 = 2 \]
\[ x = 1 \text{ or } x = 5 \]
x-intercepts: (1,0) and (5,0)

\[ y = (0-3)^2-4 \]
\[ y = 9-4=5 \]
y-intercept: (0,5)

Domain: \[ (-\infty, \infty) \]
Range: \[ (-\infty, 5] \]

Graphing Quadratic Functions in the Form \( f(x)=ax^2+bx+c \)

The Vertex of a Parabola Whose Equation Is \( f(x) = ax^2 + bx + c \)

Consider the parabola defined by the quadratic function \( f(x) = ax^2 + bx + c \).

The parabola’s vertex is \( \left( -\frac{b}{2a}, f\left( -\frac{b}{2a} \right) \right) \).

Graphing Quadratic Functions with Equations in the Form \( f(x) = ax^2 + bx + c \)

1. Determine whether the parabola opens upward or downward. If \( a > 0 \), it opens upward. If \( a < 0 \), it opens downward.
2. Determine the vertex of the parabola. The vertex is \( \left( -\frac{b}{2a}, f\left( -\frac{b}{2a} \right) \right) \).
3. Find any \( x \)-intercepts by solving \( f(x) = 0 \). The real solutions of \( ax^2 + bx + c = 0 \) are the \( x \)-intercepts.
4. Find the \( y \)-intercept by computing \( f(0) \). Because \( f(0) = c \) (the constant term in the function’s equation), the \( y \)-intercept is \( c \) and the parabola passes through \((0, c)\).
5. Plot the intercepts, the vertex, and additional points as necessary. Connect these points with a smooth curve.
Using the form \( f(x)=ax^2+bx+c \)

\[ f(x) = x^2 + 2x + 1 \quad a=1, \ b=2, \ c=1 \]

Vertex \( \left( \frac{-b}{2a}, f\left(\frac{-b}{2a}\right)\right) \) \( \quad \frac{-b}{2a} = \frac{-2}{2 \times 1} = -1 \)

\[ f(-1) = (-1)^2 + 2(-1) + 1 = 0 \quad V(-1,0) \]

Axis of symmetry \( x=-1 \)

Finding x intercept

\[
0 = x^2 + 2x + 1 \\
0 = (x + 1)(x + 1) \\
x + 1 = 0 \\
x = -1 \quad (-1,0) \text{ x intercept}
\]

Finding y intercept

\[
y = 0^2 + 2 \times 0 + 1 \\
y = 1 \quad (0,1) \text{ y intercept}
\]

Example

Find the vertex of the function \( f(x)=-x^2-3x+7 \)

\[ a = -1, \ b = -3, \ c = 7 \]

\[
x = \frac{-b}{2a} = \frac{-(-3)}{2(-1)} = \frac{3}{2}
\]

\[
y = f\left(\frac{-3}{2}\right) = -\left(\frac{-3}{2}\right)^2 - 3\left(\frac{-3}{2}\right) + 7 = -\frac{9}{4} + \frac{9}{2} + 7 = -\frac{9}{4} + \frac{18}{4} + \frac{28}{4} = \frac{37}{4}
\]

The vertex is \( \left(\frac{-3}{2}, \frac{37}{4}\right) \).
The domain of any quadratic function includes all real numbers. If the vertex is the graph’s highest point, the range includes all real numbers at or below the y-coordinate of the vertex. If the vertex is the graph’s lowest point, the range includes all real numbers at or above the y-coordinate of the vertex.

Graph the function $f(x) = -x^2 - 3x + 7$. Use the graph to identify the domain and range.
Objective 3: Determine a quadratic function’s minimum or maximum value.

**Minimum and Maximum: Quadratic Functions**

Consider the quadratic function \( f(x) = ax^2 + bx + c \).

1. If \( a > 0 \), then \( f \) has a minimum that occurs at \( x = -\frac{b}{2a} \). This minimum value is \( f\left(-\frac{b}{2a}\right) \).

2. If \( a < 0 \), then \( f \) has a maximum that occurs at \( x = -\frac{b}{2a} \). This maximum value is \( f\left(-\frac{b}{2a}\right) \).

In each case, the value of \( x \) gives the location of the minimum or maximum value. The value of \( y \), or \( f\left(-\frac{b}{2a}\right) \), gives that minimum or maximum value.

**Example**

For the function \( f(x) = -3x^2 + 2x - 5 \)

Without graphing determine whether it has a minimum or maximum and find it.

Identify the function’s domain and range.
Graphing Calculator – Finding the Minimum or Maximum

Input the equation into Y=
Go to 2nd Trace to get Calculate. Choose #4 for Maximum or #3 for Minimum.

Move your cursor to the left (left bound) of the relative minimum or maximum point that you want to know the vertex for. Press Enter.

Then move your cursor to the other side of the vertex – the right side of the vertex when it asks for the right bound. Press Enter.

When it asks to guess, you can or simply press Enter. The next screen will show you the coordinates of the maximum or minimum.

Objective 4: Solve problems involving a quadratic function’s minimum or maximum value.

Strategy for Solving Problems Involving Maximizing or Minimizing Quadratic Functions

1. Read the problem carefully and decide which quantity is to be maximized or minimized.
2. Use the conditions of the problem to express the quantity as a function in one variable.
3. Rewrite the function in the form \( f(x) = ax^2 + bx + c \).
4. Calculate \( \frac{-b}{2a} \). If \( a > 0 \), \( f \) has a minimum at \( x = \frac{-b}{2a} \). This minimum value is \( f\left(\frac{-b}{2a}\right) \). If \( a < 0 \), \( f \) has a maximum at \( x = \frac{-b}{2a} \). This maximum value is \( f\left(\frac{-b}{2a}\right) \).
5. Answer the question posed in the problem.
Example

You have 64 yards of fencing to enclose a rectangular region. Find the dimensions of the rectangle that maximize the enclosed area. What is the maximum area?