1.8 Inverse Functions

Objective 1: Verify inverse functions.

**Definition of the Inverse of a Function**

Let $f$ and $g$ be two functions such that

\[
 f(g(x)) = x \quad \text{for every } x \text{ in the domain of } g
\]

and

\[
 g(f(x)) = x \quad \text{for every } x \text{ in the domain of } f.
\]

The function $g$ is the inverse of the function $f$ and is denoted by $f^{-1}$ (read “$f$-inverse”). Thus, $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$. The domain of $f$ is equal to the range of $f^{-1}$, and vice versa.

The function $f$ is a set of ordered pairs, $(x,y)$, then the changes produced by $f$ can be “undone” by reversing components of all the ordered pairs. The resulting relation $(y,x)$, may or may not be a function. Inverse functions have a special “undoing” relationship.

The notation $f^{-1}$ represents the inverse function of $f$. The -1 is not an exponent.

The notation $f^{-1}$ does not mean $\frac{1}{f}$. 

Relations, Functions and one-to-one Functions

One-to-one Functions are a subset of Functions. They are special functions where for every $x$, there is one $y$, and for every $y$, there is one $x$.

Reminder: The definition of function is, for every $x$ there is only one $y$.

Inverse Functions are 1:1

Find $f(g(x))$ and $g(f(x))$ and determine whether each pair of functions $f$ and $g$ are inverses of each other.

$f(x) = 4x + 9$ and $g(x) = \frac{x - 9}{4}$
Objective 2: Find the inverse of a function.

Finding the Inverse of a Function
The equation for the inverse of a function \( f \) can be found as follows:
1. Replace \( f(x) \) with \( y \) in the equation for \( f(x) \).
2. Interchange \( x \) and \( y \).
3. Solve for \( y \). If this equation does not define \( y \) as a function of \( x \), the function \( f \) does not have an inverse function and this procedure ends. If this equation does define \( y \) as a function of \( x \), the function \( f \) has an inverse function.
4. If \( f \) has an inverse function, replace \( y \) in step 3 by \( f^{-1}(x) \). We can verify our result by showing that \( f(f^{-1}(x)) = x \) and \( f^{-1}(f(x)) = x \).

How to Find an Inverse Function

Find the inverse function of \( f(x) \).
\( f(x) = x^2 - 1, \ x \geq 0 \)
1. Replace \( f(x) \) with \( y \): \( y = x^2 - 1 \)
2. Interchange \( x \) and \( y \): \( x = y^2 - 1 \)
3. Solve for \( y \): \( x + 1 = y^2 \)
   \( \sqrt{x + 1} = y \)
4. Replace \( y \) with \( f^{-1}(x) \): \( f^{-1}(x) = \sqrt{x + 1} \)
Example

Find the inverse of $f(x)=7x-1$

Example

Find the inverse of $f(x)=x^3 + 4$
Example

Find the inverse of \( f(x) = \frac{3}{x} - 5 \)

Objective 3: Use the horizontal line test to determine if a function has an inverse function.

The Horizontal Line Test for Inverse Functions
A function \( f \) has an inverse that is a function, \( f^{-1} \), if there is no horizontal line that intersects the graph of the function \( f \) at more than one point.
b and c are not one-to-one functions because they don’t pass the horizontal line test.

Example

Graph the following function and tell whether it has an inverse function or not.

\[ f(x) = \sqrt{x - 3} \]
Example

Graph the following function and tell whether it has an inverse function or not.

\[ f(x) = |x - 1| \]

There is a relationship between the graph of a one-to-one function, \( f \), and its inverse \( f^{-1} \). Because inverse functions have ordered pairs with the coordinates interchanged, if the point \((a, b)\) is on the graph of \( f \) then the point \((b, a)\) is on the graph of \( f^{-1} \). The points \((a, b)\) and \((b, a)\) are symmetric with respect to the line \( y=x \). Thus graph of \( f^{-1} \) is a reflection of the graph of \( f \) about the line \( y=x \).
A function and its inverse graphed on the same axis.

Example

If this function has an inverse function, then graph it's inverse on the same graph.

\[ f(x) = \sqrt{x - 3} \]
Example
If this function has an inverse function, then graph it’s inverse on the same graph. $f(x) = x^3$

Objective 5: Find the inverse of a function and graph both functions on the same axes.
   a. Find an equation for $f^{-1}(x)$.
   b. Graph $f$ and $f^{-1}$ in the same rectangular coordinate system.
   c. Use interval notation to give the domain and the range of $f$ and $f^{-1}$.

$f(x) = x^2 - 1, \quad x \leq 0$
Use a graphing calculator to graph the function. Use the graph to determine whether the function has an inverse that is a function (that is whether the function is one-to-one).

\[ f(x) = x^2 - 1 \]

\[ f(x) = (x - 1)^3 \]