Probability
Assignment of Probabilities

Example: Probability of an Outcome
- Toss an unbiased coin and observe the side that faces upward. Determine the probability distribution for this experiment.
- Since the coin is unbiased, each outcome is equally likely to occur.

<table>
<thead>
<tr>
<th>Outcome</th>
<th>Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>Heads</td>
<td>½</td>
</tr>
<tr>
<td>Tails</td>
<td>½</td>
</tr>
</tbody>
</table>

Example: Experimental Probability
Traffic engineers measure the volume of traffic on a major highway during the rush hour.
Generate a probability distribution using the data generated over 300 consecutive weekdays.

<table>
<thead>
<tr>
<th>Number of cars observed</th>
<th>Frequency observed</th>
</tr>
</thead>
<tbody>
<tr>
<td>≤ 1000</td>
<td>30</td>
</tr>
<tr>
<td>1001–3000</td>
<td>45</td>
</tr>
<tr>
<td>3001–5000</td>
<td>135</td>
</tr>
<tr>
<td>5001–7000</td>
<td>75</td>
</tr>
<tr>
<td>&gt; 7000</td>
<td>15</td>
</tr>
</tbody>
</table>

Example: Experimental Probability
We will use the experimental probability for the distribution.

<table>
<thead>
<tr>
<th>Outcome</th>
<th>Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>$s_1$</td>
<td>$\frac{30}{300} = .10$</td>
</tr>
<tr>
<td>$s_2$</td>
<td>$\frac{45}{300} = .15$</td>
</tr>
<tr>
<td>$s_3$</td>
<td>$\frac{135}{300} = .45$</td>
</tr>
<tr>
<td>$s_4$</td>
<td>$\frac{75}{300} = .25$</td>
</tr>
<tr>
<td>$s_5$</td>
<td>$\frac{15}{300} = .05$</td>
</tr>
</tbody>
</table>

Probability of an Outcome
- Let a sample space $S$ consist of a finite number of outcomes $s_1, s_2, \ldots, s_N$. To each outcome we associate a number, called the probability of the outcome, which represents the relative likelihood that the outcome will occur.
- A chart showing the outcomes and the assigned probability is called the probability distribution for the experiment.

Experimental Probability
- Let a sample space $S$ consist of a finite number of outcomes $s_1, s_2, \ldots, s_N$. The relative frequency, or experimental probability, of each outcome is calculated after many trials.
- The experimental probability could be different for a different set of trials and different from the probability of the events.
Fundamental Properties of Probabilities

- Let an experiment have outcomes $s_1, s_2, \ldots, s_N$ with probabilities $p_1, p_2, \ldots, p_N$.
  Then the numbers $p_1, p_2, \ldots, p_N$ must satisfy:
  
  **Fundamental Property 1**
  
  Each of the numbers $p_1, p_2, \ldots, p_N$ is between 0 and 1;

  **Fundamental Property 2**
  
  $p_1 + p_2 + \ldots + p_N = 1$.

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Example: Fundamental Properties

Verify the fundamental properties for the following distribution.

<table>
<thead>
<tr>
<th>Outcome</th>
<th>Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>$s_1$</td>
<td>$\frac{30}{300} = .10$</td>
</tr>
<tr>
<td>$s_2$</td>
<td>$\frac{45}{300} = .15$</td>
</tr>
<tr>
<td>$s_3$</td>
<td>$\frac{135}{300} = .45$</td>
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<tr>
<td>$s_4$</td>
<td>$\frac{75}{300} = .25$</td>
</tr>
<tr>
<td>$s_5$</td>
<td>$\frac{15}{300} = .05$</td>
</tr>
</tbody>
</table>

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Addition Principle

- **Addition Principle**
  
  Suppose that an event $E$ consists of the finite number of outcomes $s, t, u, \ldots, z$.
  That is $E = \{s, t, u, \ldots, z\}$.
  
  Then
  
  $\Pr(E) = \Pr(s) + \Pr(t) + \Pr(u) + \ldots + \Pr(z)$,
  where $\Pr(A)$ is the probability of event $A$.

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Example: Addition Principle

- Suppose that we toss a red die and a green die and observe the numbers on the sides that face upward.
  
  a.) Calculate the probabilities of the elementary events.
  b.) Calculate the probability that the two dice show the same number.

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Inclusion-Exclusion Principle

- Let $E$ and $F$ be any events. Then
  
  $\Pr(E \cup F) = \Pr(E) + \Pr(F) - \Pr(E \cap F)$.
  
  If $E$ and $F$ are mutually exclusive, then
  
  $\Pr(E \cup F) = \Pr(E) + \Pr(F)$.

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Example: Inclusion-Exclusion Principle

A factory needs two raw materials. The probability of not having an adequate supply of material $A$ is .05 and the probability of not having an adequate supply of material $B$ is .03. A study determines that the probability of a shortage of both materials is .01. What proportion of the time will the factory not be able to operate from lack of materials?
Odds

If the odds in favor of an event $E$ are $a$ to $b$, then

$$Pr(E) = \frac{a}{a+b} \text{ and } Pr(E') = \frac{b}{a+b}.$$ 

On average, for every $a+b$ trials, $E$ will occur $a$ times and $E$ will not occur $b$ times.

Example: Odds

In the two dice problem, what are the odds of rolling a pair with the same number on the faces?

The probability of obtaining a pair with the same number on the faces is $1/6$. The probability of not obtaining a pair with the same number on the faces is $5/6$. The odds are

$$\frac{1/6}{5/6} = \frac{1}{5} \text{ or } 1 \text{ to } 5.$$ 

Example

An experiment consists of tossing a coin two times and observing the sequence of heads and tails. Each of the four outcomes has the same probability of occurring.

a.) What is the probability that “HH” is the outcome?

b.) What is the probability of the event “at least one head”?

Example

Which of the following probabilities are feasible for an experiment having sample space $\{s_1, s_2, s_3\}$?

a.) $Pr(s_1) = .4$, $Pr(s_2) = .4$, $Pr(s_3) = .4$

b.) $Pr(s_1) = .5$, $Pr(s_2) = .7$, $Pr(s_3) = -.2$

c.) $Pr(s_1) = 2$, $Pr(s_2) = 1$, $Pr(s_3) = \frac{1}{2}$

d.) $Pr(s_1) = .25$, $Pr(s_2) = .5$, $Pr(s_3) = .25$

Example

An experiment consists of selecting a number at random from the set of numbers $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$. Find the probability that the number selected is:

a.) less than 4

b.) odd

c.) less than 4 or odd

Example

An experiment with the following outcomes is described by the probability table at right. Let $E = \{s_1, s_2\}$ and $F = \{s_3, s_5, s_6\}$. Determine each below.

a.) $Pr(E)$ and $Pr(F)$.

b.) $Pr(E')$

c.) $Pr(E \cap F)$

d.) $Pr(E \cup F)$

<table>
<thead>
<tr>
<th>Outcome</th>
<th>Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>S1</td>
<td>.05</td>
</tr>
<tr>
<td>S2</td>
<td>.25</td>
</tr>
<tr>
<td>S3</td>
<td>.05</td>
</tr>
<tr>
<td>S4</td>
<td>.01</td>
</tr>
<tr>
<td>S5</td>
<td>.63</td>
</tr>
<tr>
<td>S6</td>
<td>.01</td>
</tr>
</tbody>
</table>
Example

Let $E$ and $F$ be events for which $\Pr(E) = .4$, $\Pr(F) = .5$, and $\Pr(E \cap F^{'}) = .3$.

Find:

a.) $\Pr(E \cap F)$

b.) $\Pr(E \cup F)$

Example

- In poker, the probability of being dealt a hand containing a pair of jacks or better is about $1/6$. What are the corresponding odds?

Example

- The odds of Americans living in the state where they are born is 16 to 9. What is the probability that an American selected at random lives in his or her birth state?

Example

- Four people are running for class president: Liz, Sam, Sue, and Tom. The probabilities of Sam, Sue, and Tom winning are .18, .23, and .31, respectively.

  a.) What is the probability of Liz winning?
  b.) What is the probability that a boy wins?
  c.) What is the probability that Tom loses?
  d.) What are the odds that Sue loses?
  e.) What are the odds that a girl wins?
  f.) What are the odds that Sam wins?

Example

- A realtor analyzes the office’s sales over the past two years. She sorts the sales by the ages of the house at the time of the sale.

  a.) Determine the probability that a sale chosen at random is a house between five and six years old.
  b.) What are the odds that the house is less than seven years old?

<table>
<thead>
<tr>
<th>Age</th>
<th># Houses Sold</th>
</tr>
</thead>
<tbody>
<tr>
<td>1-2</td>
<td>1200</td>
</tr>
<tr>
<td>3-4</td>
<td>1570</td>
</tr>
<tr>
<td>5-6</td>
<td>1600</td>
</tr>
<tr>
<td>7-8</td>
<td>1520</td>
</tr>
<tr>
<td>9-10</td>
<td>1480</td>
</tr>
</tbody>
</table>