8.5 POWER SERIES

A power series is a series of the form

\[ \sum_{n=1}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots \]

where \( x \) is a variable and the \( c_n \)'s are constants called the coefficients of the series.

DEFINITION: POWER SERIES

A power series has the general form

\[ \sum_{n=0}^{\infty} c_n (x - a)^n = c_0 + c_1(x - a) + c_2(x - a)^2 + \cdots \]

where \( a \) and \( c_n \) are real numbers, and \( x \) is a variable.

The \( c_n \)'s are the coefficients of the series and \( a \) is the center of the power series.

The set of \( x \) values for which the series converges is the interval of convergence. The radius of convergence of the series, denoted \( R \), is the distance from the center of the series to the boundary of the interval of convergence.
Three possibilities for power series $\sum_{n=0}^{\infty} c_n (x - a)^n$

i) The series converges only when $x = a$. (in this case, $R = 0$).

ii) The series converges for all $x$. (in this case, $R = \infty$)

iii) There is a positive number $R$ such that the series converges if $|x - a| < R$ and diverges when $|x - a| > R$.

$R$ is the radius of convergence of the power series.

Find the interval and radius of convergence for each power series. Sketch the interval and radius on a number line.

**a) $\sum_{n=0}^{\infty} \frac{x^n}{n!}$**

Ans. Inter: $(-\infty, \infty)$; $R = \infty$

center of power series

**b) $\sum_{n=0}^{\infty} \frac{(-1)^n (x - 2)^n}{4^n}$**

Ans. Inter: (-2, 6); $R = 4$

Center of power series
c) \[ \sum_{n=1}^{\infty} n!x^n \]

Ans. By the Ratio Test, this series is divergent. The only way for \( r < 1 \), and therefore convergent, is to take \( r = 0 \). In this case the power series has a value of 0. The interval of radius and the radius is the single point \( x = 0 \).

\[ 0 \]

Interval of convergence \( [0] \)
Radius of convergence \( R = 0 \)

\[ \sum_{n=1}^{\infty} \frac{(x-2)^{n+1}}{\sqrt{n}} \]

Ans. Int of convergence: \( [1, 3) \)
Radius of convergence \( R = 1 \)